

# Chiral Determinant Formulae and Subsingular Vectors for the N=2 Superconformal Algebras

Beatriz Gato-Rivera<sup>a,b</sup> and Jose Ignacio Rosado<sup>a</sup>

<sup>a</sup> *Instituto de Matemáticas y Física Fundamental, CSIC,  
Serrano 123, Madrid 28006, Spain \**

<sup>b</sup> *NIKHEF, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands*

## ABSTRACT

We derive conjectures for the N=2 “chiral” determinant formulae of the Topological algebra, the Antiperiodic NS algebra, and the Periodic R algebra, corresponding to incomplete Verma modules built on chiral topological primaries, chiral and antichiral NS primaries, and Ramond ground states, respectively. Our method is based on the analysis of the singular vectors in chiral Verma modules and their spectral flow symmetries, together with some computer exploration and some consistency checks. In addition, and as a consequence, we uncover the existence of subsingular vectors in these algebras, giving examples (subsingular vectors are non-highest-weight null vectors which are not descendants of any highest-weight singular vectors).

October 1997

---

\*e-mail address: bgato@pinar1.csic.es

# 1 Introduction and Notation

The  $N=2$  Superconformal algebras provide the symmetries underlying the  $N=2$  strings [1] [2] [3] [4]. In addition, the topological version of the algebra is realized in the world-sheet of the bosonic string [5], as well as in the world-sheet of the superstrings [6]. Various aspects concerning singular vectors of the  $N=2$  Superconformal algebras in chiral Verma modules have been studied in several papers during the last few years, most of them involving the Topological algebra. For example, in [5] and [7] it was shown that the uncharged BRST-invariant singular vectors, in the “mirror bosonic string” realization (KM) of the Topological algebra, are related to the Virasoro constraints on the KP  $\tau$ -function. In [8] an isomorphism was uncovered between the uncharged BRST-invariant singular vectors and the singular vectors of the  $\mathfrak{sl}(2)$  Kac-Moody algebra. Some properties of the topological singular vectors in the DDK and KM realizations were analyzed in [9]. In [10] the complete set of topological singular vectors at level 2 (four types) was written down, together with the spectral flow automorphism  $\mathcal{A}$  which transforms all kinds of topological singular vectors into each other. In [11] the family structure of the topological singular vectors was analyzed (not only in chiral Verma modules but also in complete Verma modules, where there are thirty-three different types of topological singular vectors).

In all of those papers some explicit examples of singular vectors in chiral Verma modules were written down, ranging from level 1 until level 4. In addition, in [8] a general formula was given for the spectrum of  $U(1)$  charges corresponding to the topological chiral Verma modules which contain uncharged BRST-invariant singular vectors. Although the formula fitted with the known data, a proof or derivation was lacking\*. Similarly, general formulae for the spectrum of conformal weights and/or  $U(1)$  charges for chiral Verma modules of the Antiperiodic NS algebra which contain singular vectors are absent, and the same is true for the Verma modules of the Periodic R algebra built on the Ramond ground states. These spectra, which cannot be obtained directly from the roots of the  $N=2$  determinant formulae [12] [13] [14], since these only apply to complete Verma modules without constraints, can be viewed rather as the roots of the “chiral” determinant formulae of the  $N=2$  Superconformal algebras (the Ramond ground states are directly related to the chiral NS primary states via the spectral flows).

In this paper we present conjectures for the  $N=2$  “chiral” determinant formulae cor-

---

\*In addition that formula, eq. (3.1) in ref. [8], was presented not as a conjecture but as a straightforward derivation from the determinant formula of the  $N=2$  Antiperiodic NS algebra, in particular from the uncharged series in ref. [12], which is not the case since the determinant formulae does not apply to incomplete Verma modules with constraints. The fact that the formula given in ref. [8] cannot be derived from the determinant formula in ref. [12], as claimed by the author, is precisely what leads to the discovery of subsingular vectors, as we will see.

responding to the chiral Verma modules of the Topological algebra, the chiral Verma modules of the Antiperiodic NS algebra, and the Verma modules of the Periodic R algebra built on the Ramond ground states. In the absence of rigorous proofs, we have checked our results from level  $1/2$  to level 4 and, in addition, we provide some consistency checks.

We have proceeded in two steps. First we have derived conjectures for the roots of the  $N=2$  chiral determinants, using some properties of the singular vectors in chiral Verma modules, together with the ansatz that the roots of the  $N=2$  chiral determinants are contained in the set of roots of the  $N=2$  determinants, and in the simplest possible way, in addition. Second, we have written down the  $N=2$  chiral determinant formulae using the conjectures for the roots, some computer exploration, and the consistency checks. Our conjectures for the roots of the  $N=2$  chiral determinants imply that there exist subsingular vectors, *i.e.* singular vectors in the chiral Verma modules which are not singular in the complete Verma modules, where they are non-highest weight null vectors not descendant of any singular vectors. To fully understand this issue [28] one has to take into account that the chiral Verma modules are nothing but the quotient of complete Verma modules by submodules generated by singular vectors.

This paper is organized as follows. In section 2 we first explain the family structure of the singular vectors of the Topological algebra built on chiral topological primaries. Then we derive the direct relation between these topological singular vectors and the singular vectors of the Antiperiodic NS algebra built on chiral primaries. This relation implies, using the family structure of the topological singular vectors, that the charged and uncharged NS singular vectors built on chiral primaries must come in pairs, although in different Verma modules related to each other by the spectral flow automorphism of the Topological algebra [10].

We use this result in section 3 to derive conjectures for the spectrum of  $U(1)$  charges  $\mathbf{h}$  (and conformal weights  $\Delta$ ) corresponding to the chiral Verma modules (NS and topological) which contain singular vectors. These spectra are the roots of the  $N=2$  chiral determinant formulae. To derive the conjectures we also make the ansatz that the roots of the  $N=2$  chiral determinants coincide with the roots of the  $N=2$  determinants (*i.e.* for complete Verma modules) specialized to the values of (or relations between) the conformal weight  $\Delta$  and  $U(1)$  charge  $\mathbf{h}$  which occur in chiral Verma modules. Namely,  $\Delta = 0$  for the case of the Topological algebra, and  $\Delta = \pm \frac{\mathbf{h}}{2}$  for the case of the Antiperiodic NS algebra (+ for chiral representations and  $-$  for antichiral representations). Actually we only need to work out the ansatz for one of the algebras since the ansatz for the other algebra follows automatically via the relations between the corresponding singular vectors. We have chosen the Antiperiodic NS algebra for this purpose. The fact that charged and uncharged singular vectors come in pairs then implies that half of the zeroes of the quadratic vanishing surface  $f_{r,s}^A = 0$ , for every pair  $(r, s)$ , correspond to uncharged

singular vectors at level  $\frac{rs}{2}$ , and the other half correspond to charged singular vectors at level  $\frac{r(s+2)-1}{2}$  (in addition to the zeroes of the vanishing plane  $g_k^A = 0$ ). This contrasts sharply with the spectra of singular vectors for complete Verma modules, for which all the zeroes of the quadratic vanishing surface  $f_{r,s}^A = 0$  correspond to uncharged singular vectors at level  $\frac{rs}{2}$ , and implies that the “new” charged singular vectors which appear in the chiral Verma modules are subsingular vectors.

In section 4 we derive the analogous conjecture for the Periodic R algebra, that is the spectrum of U(1) charges  $\mathbf{h}$  corresponding to the R Verma modules, built on the Ramond ground states (for which  $\Delta = \frac{c}{24}$ ), which contain singular vectors. For this we only need to use the spectral flows, with the appropriate parameters which map singular vectors of the Antiperiodic NS algebra, built on chiral primaries, into singular vectors of the Periodic R algebra, built on the Ramond ground states.

In section 5 we finally write down expressions for the N=2 chiral determinant formulae, using the conjectures for the roots together with some computer investigation and some consistency checks. Section 6 is devoted to conclusions and final remarks.

Finally, in the Appendix we analyze thoroughly the NS singular vectors at levels 1 and  $\frac{3}{2}$ . We write down all the equations resulting from the highest weight conditions, showing that these equations, and therefore their solutions, are different when imposing or not chirality on the primary states on which the singular vectors are built. Namely, when the primary state is non-chiral all the solutions of the quadratic vanishing surface  $f_{1,2}^A = 0$  correspond to level 1 uncharged singular vectors. When the primary state is chiral, however, half of the solutions of  $f_{1,2}^A = 0$  specialized to the cases  $\Delta = \pm \frac{\mathbf{h}}{2}$  correspond to level  $\frac{3}{2}$  charged singular vectors. We show that these charged singular vectors become non-highest weight null vectors, once the chirality on the primary state is switched off, which are not descendants of any h.w. singular vector, *i.e.* they are subsingular vectors.

## Notation

*Primary states* denote non-singular highest weight (h.w.) vectors.

*Null vectors* are zero-norm states (not necessarily h.w.).

*Singular vectors* are h.w. null vectors, equivalently h.w. descendant states.

*Subsingular vectors* are non-h.w. null vectors not descendants of any singular vectors, they become singular (*i.e.* h.w.) in the quotient of the Verma module by a submodule generated by singular vectors.

The Antiperiodic NS algebra will be denoted as the *NS algebra*. The chiral and antichiral primary states and Verma modules will be denoted simply as *chiral*, unless otherwise indicated.

The Periodic R algebra will be denoted as the *R algebra*.

The singular vectors of the Topological algebra will be denoted as *topological singular vectors*  $|\chi_T\rangle$ .

The singular vectors of the NS algebra will be denoted as *NS singular vectors*  $|\chi_{NS}\rangle$ .

The singular vectors of the R algebra will be denoted as *R singular vectors*  $|\chi_R\rangle$ .

## 2 Some Properties of Singular Vectors in Chiral Verma Modules

### 2.1 Singular Vectors of the Topological Algebra in Chiral Verma Modules

*The Topological Algebra.*— The algebra obtained by applying the topological twists [15], [16] on the N=2 Superconformal algebra reads [17]

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, & m, n \in \mathbf{Z}. \quad (2.1) \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \frac{c}{6}(m^2+m)\delta_{m+n,0}, \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2+m)\delta_{m+n,0},
\end{aligned}$$

where  $\mathcal{L}_m$  and  $\mathcal{H}_m$  are the bosonic generators corresponding to the energy momentum tensor (Virasoro generators) and the topological  $U(1)$  current respectively, while  $\mathcal{Q}_m$  and  $\mathcal{G}_m$  are the fermionic generators corresponding to the BRST current and the spin-2 fermionic current respectively. The “topological central charge”  $c$  is the central charge of the untwisted N=2 Superconformal algebra. This algebra is topological because the Virasoro generators can be expressed as  $\mathcal{L}_m = \frac{1}{2}\{\mathcal{G}_m, \mathcal{Q}_0\}$ , where  $\mathcal{Q}_0$  is the BRST charge. This implies, as is well known, that the correlators of fields do not depend on the metric.

*The Topological Twists.*— The Topological algebra is satisfied by the two sets of topological generators

$$\begin{aligned}
\mathcal{L}_m^{(1)} &= L_m + \frac{1}{2}(m+1)H_m, \\
\mathcal{H}_m^{(1)} &= H_m, \\
\mathcal{G}_m^{(1)} &= G_{m+\frac{1}{2}}^+, \quad \mathcal{Q}_m^{(1)} = G_{m-\frac{1}{2}}^-,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\mathcal{L}_m^{(2)} &= L_m - \frac{1}{2}(m+1)H_m, \\
\mathcal{H}_m^{(2)} &= -H_m, \\
\mathcal{G}_m^{(2)} &= G_{m+\frac{1}{2}}^-, \quad \mathcal{Q}_m^{(2)} = G_{m-\frac{1}{2}}^+,
\end{aligned} \tag{2.3}$$

corresponding to the two possible twistings of the superconformal generators  $L_m, H_m, G_m^+$  and  $G_m^-$ . The topological twists, which we denote as  $T_{W1}$  and  $T_{W2}$ , are mirrored under the interchange  $H_m \leftrightarrow -H_m, G_r^+ \leftrightarrow G_r^-$ , as one can see.

*Chiral Topological Primaries.*— Of special importance are the topological primary states annihilated by both  $\mathcal{G}_0$  and  $\mathcal{Q}_0$ , denoted as “chiral”. The anticommutator  $\{\mathcal{G}_0, \mathcal{Q}_0\} = 2\mathcal{L}_0$  shows that these states have zero conformal weight  $\Delta$ , therefore their only quantum number is their U(1) charge  $\mathbf{h}$  ( $\mathcal{H}_0|\Phi\rangle = \mathbf{h}|\Phi\rangle$ ). The secondary states built on chiral primaries have positive conformal weight  $\Delta = l > 0$ , as a result, where  $l$  is the level of the state. This anticommutator also shows that any secondary state with conformal weight  $\Delta \neq 0$  can be decomposed into a  $\mathcal{Q}_0$ -closed state and a  $\mathcal{G}_0$ -closed state, and also implies that a  $\mathcal{Q}_0$ -closed secondary state with  $\Delta \neq 0$  is  $\mathcal{Q}_0$ -exact as well (and similarly with  $\mathcal{G}_0$ ). Therefore, the chiral primaries are physical states (BRST-closed but not BRST-exact), whereas the secondary states built on chiral primaries are not physical.

Regarding the twists (2.2) and (2.3), a key observation is that  $(G_{1/2}^+, G_{-1/2}^-)$  results in  $(\mathcal{G}_0^{(1)}, \mathcal{Q}_0^{(1)})$  and  $(G_{1/2}^-, G_{-1/2}^+)$  gives  $(\mathcal{G}_0^{(2)}, \mathcal{Q}_0^{(2)})$ . This brings about two important consequences. First, the chiral topological primaries  $|\Phi\rangle^{(1)}$  and  $|\Phi\rangle^{(2)}$  correspond to the antichiral primaries (*i.e.*  $G_{-1/2}^-|\Phi\rangle^{(1)} = 0$ ) and to the chiral primaries (*i.e.*  $G_{-1/2}^+|\Phi\rangle^{(2)} = 0$ ) of the NS algebra, respectively. Second, one of the highest weight (h.w.) conditions  $G_{1/2}^\pm|\chi_{NS}\rangle = 0$ , of the NS algebra, read  $\mathcal{G}_0|\chi_T\rangle = 0$  after the corresponding twistings. Therefore, any h.w. state (primary or secondary) of the NS algebra results in a topological state annihilated by  $\mathcal{G}_0$ , under the twistings. We will discuss in detail this issue in next subsection.

*Topological Secondaries.*— A topological secondary, or descendant, state can also be labeled by its level  $l$  (the conformal weight for states built on chiral topological primaries), and its U(1) charge  $(\mathbf{h} + q)$  (the  $\mathcal{H}_0$ -eigenvalue). It is convenient to split the total U(1) charge into two pieces: the U(1) charge  $\mathbf{h}$  of the primary state on which the descendant is built, thus labeling the corresponding Verma module  $V_T(\mathbf{h})$ , and the “relative” U(1)

charge  $q$ , corresponding to the operator acting on the primary state, given by the number of  $\mathcal{G}$  modes minus the number of  $\mathcal{Q}$  modes in each term. We will denote the  $\mathcal{Q}_0$ -closed and  $\mathcal{G}_0$ -closed topological secondary states as  $|\chi_T\rangle^{(q)Q}$  and  $|\chi_T\rangle^{(q)G}$  respectively (notice that  $(q)$  refers to the relative  $U(1)$  charge of the state). Usually we will also indicate the level  $l$  and/or the Verma module  $\mathbf{h}$ .

*Topological Singular Vectors on Chiral Primaries.*— It turns out that the topological singular vectors built on chiral primaries come only in four types<sup>†</sup> [11]:  $|\chi_T\rangle^{(0)G}$ ,  $|\chi_T\rangle^{(0)Q}$ ,  $|\chi_T\rangle^{(1)G}$  and  $|\chi_T\rangle^{(-1)Q}$ . These four types of singular vectors can be mapped into each other by using  $\mathcal{G}_0$ ,  $\mathcal{Q}_0$  and the spectral flow automorphism of the Topological algebra, denoted by  $\mathcal{A}$ . The action of  $\mathcal{G}_0$  and  $\mathcal{Q}_0$  results in singular vectors in the same Verma module (by definition):

$$\begin{aligned}\mathcal{Q}_0|\chi_T\rangle_{l,\mathbf{h}}^{(0)G} &\rightarrow |\chi_T\rangle_{l,\mathbf{h}}^{(-1)Q}, & \mathcal{G}_0|\chi_T\rangle_{l,\mathbf{h}}^{(-1)Q} &\rightarrow |\chi_T\rangle_{l,\mathbf{h}}^{(0)G}, \\ \mathcal{Q}_0|\chi_T\rangle_{l,\mathbf{h}}^{(1)G} &\rightarrow |\chi_T\rangle_{l,\mathbf{h}}^{(0)Q}, & \mathcal{G}_0|\chi_T\rangle_{l,\mathbf{h}}^{(0)Q} &\rightarrow |\chi_T\rangle_{l,\mathbf{h}}^{(1)G}.\end{aligned}$$

The level of the vectors does not change under the action of  $\mathcal{G}_0$  and  $\mathcal{Q}_0$ , obviously. Since the Verma module does not change either, charged and uncharged topological singular vectors, with different BRST-invariance properties, come always in pairs in the same Verma module. Namely, singular vectors of the types  $|\chi_T\rangle^{(0)Q}$  and  $|\chi_T\rangle^{(1)G}$  are together in the same Verma module at the same level, and a similar statement holds for the singular vectors of the types  $|\chi_T\rangle^{(0)G}$  and  $|\chi_T\rangle^{(-1)Q}$ .

The spectral flow automorphism of the topological algebra, on the other hand, given by [10]

$$\begin{aligned}\mathcal{A}\mathcal{L}_m\mathcal{A} &= \mathcal{L}_m - m\mathcal{H}_m, \\ \mathcal{A}\mathcal{H}_m\mathcal{A} &= -\mathcal{H}_m - \frac{\epsilon}{3}\delta_{m,0}, \\ \mathcal{A}\mathcal{Q}_m\mathcal{A} &= \mathcal{G}_m, \\ \mathcal{A}\mathcal{G}_m\mathcal{A} &= \mathcal{Q}_m,\end{aligned}\tag{2.4}$$

with  $\mathcal{A}^{-1} = \mathcal{A}$ , changes the Verma module of the vectors as  $V_T(\mathbf{h}) \rightarrow V_T(-\mathbf{h} - \frac{\epsilon}{3})$ . In addition,  $\mathcal{A}$  reverses the relative charge as well as the BRST-invariance properties of the vectors, leaving the level invariant. Therefore the action of  $\mathcal{A}$  results in the following mappings [10]

$$\mathcal{A}|\chi_T\rangle_{l,\mathbf{h}}^{(0)G} \rightarrow |\chi_T\rangle_{l,-\mathbf{h}-\frac{\epsilon}{3}}^{(0)Q}, \quad \mathcal{A}|\chi_T\rangle_{l,\mathbf{h}}^{(-1)Q} \rightarrow |\chi_T\rangle_{l,-\mathbf{h}-\frac{\epsilon}{3}}^{(1)G}.\tag{2.5}$$

---

<sup>†</sup>A rigorous proof will be presented in a revised version of ref. [11].

*Family Structure.*— As a consequence of the mappings given by  $\mathcal{G}_0$ ,  $\mathcal{Q}_0$  and  $\mathcal{A}$ , the topological singular vectors built on chiral primaries come in families of four [11], one of each kind at the same level. Two of them, one charged and one uncharged, belong to the Verma module  $V_T(\mathbf{h})$ , whereas the other pair belong to a different Verma module  $V_T(-\mathbf{h} - \frac{\mathbf{c}}{3})$ , as the diagram shows.

$$\begin{array}{ccc}
|\chi_T\rangle_{l,\mathbf{h}}^{(0)G} & \xrightarrow{\mathcal{Q}_0} & |\chi_T\rangle_{l,\mathbf{h}}^{(-1)Q} \\
\mathcal{A} \updownarrow & & \updownarrow \mathcal{A} \\
|\chi_T\rangle_{l,-\mathbf{h}-\frac{\mathbf{c}}{3}}^{(0)Q} & \xrightarrow{\mathcal{G}_0} & |\chi_T\rangle_{l,-\mathbf{h}-\frac{\mathbf{c}}{3}}^{(1)G}
\end{array} \tag{2.6}$$

The arrows  $\mathcal{Q}_0$  and  $\mathcal{G}_0$  can be reversed using  $\mathcal{G}_0$  and  $\mathcal{Q}_0$ , respectively (up to a constant). For  $\mathbf{h} = -\frac{\mathbf{c}}{6}$  the two Verma modules related by the spectral flow automorphism coincide. Therefore, if there are singular vectors for this value of  $\mathbf{h}$  (see in section 3), they must come four by four in the same Verma module: one of each kind at the same level.

## 2.2 Untwisting the Topological Singular Vectors

The relation between the Topological algebra and the NS algebra is given by the topological twists  $T_{W1}$  (2.2) and  $T_{W2}$  (2.3). From the purely formal point of view, the Topological algebra (2.1) is simply a rewriting of the NS algebra, given by [1], [18], [20], [21]

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\mathbf{c}}{12}(m^3 - m)\delta_{m+n,0}, & [H_m, H_n] &= \frac{\mathbf{c}}{3}m\delta_{m+n,0}, \\
[L_m, G_r^\pm] &= \left(\frac{m}{2} - r\right)G_{m+r}^\pm, & [H_m, G_r^\pm] &= \pm G_{m+r}^\pm, \\
[L_m, H_n] &= -nH_{m+n} \\
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)H_{r+s} + \frac{\mathbf{c}}{3}(r^2 - \frac{1}{4})\delta_{r+s,0},
\end{aligned} \tag{2.7}$$

where the fermionic modes take half-integer values. The question naturally arises now whether or not the topological singular vectors are also a rewriting of the NS singular vectors. For the case of singular vectors built on chiral primaries the answer is that *only the  $\mathcal{G}_0$ -closed topological singular vectors transform into NS singular vectors after the untwisting*. Moreover, the resulting NS singular vectors are built on chiral primaries of the NS algebra when using the twist  $T_{W2}$  (2.3), or on antichiral primaries when using the twist  $T_{W1}$  (2.2). On the other hand, *the twisting of NS singular vectors always produces topological singular vectors annihilated by  $\mathcal{G}_0$* . They are built on chiral topological



primaries provided the NS singular vectors are built on chiral or antichiral primaries; otherwise, the topological singular vectors would be built on topological  $\mathcal{G}_0$ -closed primaries which are not chiral.

All these statements can be verified rather easily. One only needs to investigate how the highest weight (h.w.) conditions satisfied by the singular vectors get modified under the twistings or untwistings given by (2.2) and (2.3). By inspecting these it is obvious that the bosonic h.w. conditions, *i.e.*  $L_{m>0}|\chi_{NS}\rangle = H_{m>0}|\chi_{NS}\rangle = 0$ , on the one hand, and  $\mathcal{L}_{m>0}|\chi_T\rangle = \mathcal{H}_{m>0}|\chi_T\rangle = 0$ , on the other hand, are conserved under the twistings and untwistings. In other words, if the topological secondary state  $|\chi_T\rangle$  satisfies the topological bosonic h.w. conditions, then the corresponding untwisted secondary state  $|\chi_{NS}\rangle$  satisfies the NS bosonic h.w. conditions and vice-versa. With the fermionic h.w. conditions things are not so straightforward. While the h.w. conditions  $\mathcal{Q}_{m>0}|\chi_T\rangle = 0$  are converted into h.w. conditions of the type  $G_{m\geq\frac{1}{2}}^\pm|\chi_{NS}\rangle = 0$  ( $G^+$  or  $G^-$  depending on the specific twist), in both twists  $\mathcal{G}_0$  is transformed into one of the  $G_{1/2}^\pm$  modes, as we pointed out in last section. But  $G_{1/2}^\pm|\chi_{NS}\rangle = 0$  is nothing but a h.w. condition satisfied by all the NS singular vectors! As a result, the twisting of a NS singular vector always produces a topological singular vector annihilated by  $\mathcal{G}_0$ . In addition, the BRST-invariance condition  $\mathcal{Q}_0|\Phi\rangle = 0$  on the chiral topological primaries is transformed into the antichirality ( $G_{-1/2}^-|\Phi\rangle = 0$ ) and the chirality ( $G_{-1/2}^+|\Phi\rangle = 0$ ) conditions on the NS primary states, under  $T_{W1}$  and  $T_{W2}$  respectively.

Now let us analyze the transformation of the  $(\mathcal{L}_0, \mathcal{H}_0)$  eigenvalues  $(l, q + \mathbf{h})$  of the  $\mathcal{G}_0$ -closed topological singular vectors  $|\chi_T\rangle_l^{(q)G}$  into  $(L_0, H_0)$  eigenvalues  $(\Delta' + l', q' + \mathbf{h}')$  of the NS singular vectors  $|\chi_{NS}\rangle_l^{(q)}$ . In other words, given a  $\mathcal{G}_0$ -closed topological singular vector in the chiral Verma module  $V_T(\mathbf{h})$ , at level  $l$  and with relative charge  $q$ , let us determine the NS Verma module  $V_{NS}(\Delta', \mathbf{h}')$ , the level  $l'$  and the relative charge  $q'$  of the corresponding untwisted NS singular vector. Observe that  $\Delta'$  denotes the conformal weight of the primary on which the NS singular vector is built ( $\Delta = 0$  for the chiral topological primaries as we discussed before). Using  $T_{W1}$  (2.2) the U(1) charge does not change,  $\mathbf{h}' = \mathbf{h}$ , whereas  $\Delta' = -\mathbf{h}/2 = -\mathbf{h}'/2$  and the level gets modified as  $l' = l - \frac{1}{2}q$ . Therefore the  $\mathcal{G}_0$ -closed topological singular vectors of the types  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(1)G}$ , in chiral topological Verma modules, are transformed under  $T_{W1}$  into NS singular vectors of the types  $|\chi_{NS}\rangle_l^{(0)a}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$  in antichiral NS Verma modules, as indicated by the superscript  $a$ . Using  $T_{W2}$  (2.3), on the other hand, the U(1) charge reverses its sign,  $\mathbf{h}' = -\mathbf{h}$  while  $\Delta' = -\mathbf{h}/2 = \mathbf{h}'/2$  and the level gets modified, again as  $l' = l - \frac{1}{2}q$ . Therefore the  $\mathcal{G}_0$ -closed topological singular vectors of the types  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(1)G}$ , in chiral topological Verma modules, result under  $T_{W2}$  in NS singular vectors of the types  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$  in chiral NS Verma modules, as indicated by the superscript  $ch$ . Notice that the values of  $\Delta'$ , in terms of  $\mathbf{h}$ , are the same under the two untwistings.

An interesting result here is that the same charged  $\mathcal{G}_0$ -closed topological singular

vector  $|\chi_T\rangle_l^{(1)G}$ , at level  $l$ , gives rise to both charge  $q = 1$  and charge  $q = -1$  NS singular vectors at level  $l - \frac{1}{2}$ , built on antichiral primaries and chiral primaries respectively. Furthermore, since there are no other types of charged  $\mathcal{G}_0$ -closed topological singular vectors in chiral Verma modules, we deduce that *all the charged NS singular vectors built on chiral primaries have relative charge  $q = -1$ , whereas all the charged NS singular vectors built on antichiral primaries have relative charge  $q = 1$* . Moreover, the  $q = 1$  and  $q = -1$  singular vectors are mirrored under the interchange  $H_m \leftrightarrow -H_m$  (therefore  $\mathbf{h} \leftrightarrow -\mathbf{h}$ ) and  $G_r^+ \leftrightarrow G_r^-$ .

Finally, an important result is the fact that the charged and uncharged NS singular vectors  $|\chi_{NS}\rangle_l^{(0)a}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ , on the one hand, and  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ , on the other hand, must come in pairs, although in different Verma modules, since they are just the untwistings of the  $\mathcal{G}_0$ -closed topological singular vectors  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(1)G}$ , which come in pairs inside the four-member topological families described in the previous subsection. Namely, from the diagram (2.6) one deduces the one-to-one mapping  $|\chi_T\rangle_{l,-\mathbf{h}-\mathbf{c}/3}^{(1)G} = \mathcal{G}_0 \mathcal{A} |\chi_T\rangle_{l,\mathbf{h}}^{(0)G}$  between uncharged and charged  $\mathcal{G}_0$ -closed topological singular vectors in the chiral topological Verma modules  $V_T(\mathbf{h})$  and  $V_T(-\mathbf{h} - \mathbf{c}/3)$  respectively. Untwisting this mapping using  $T_{W1}$  (2.2) and  $T_{W2}$  (2.3) one obtains the one-to-one mappings

$$|\chi_{NS}\rangle_{l-\frac{1}{2},-\mathbf{h}-\frac{\mathbf{c}}{3}}^{(1)a} = G_{1/2}^+ \mathcal{A}_1 |\chi_{NS}\rangle_{l,\mathbf{h}}^{(0)a}, \quad |\chi_{NS}\rangle_{l-\frac{1}{2},\mathbf{h}+\frac{\mathbf{c}}{3}}^{(-1)ch} = G_{1/2}^- \mathcal{A}_{-1} |\chi_{NS}\rangle_{l,-\mathbf{h}}^{(0)ch} \quad (2.8)$$

between uncharged and charged NS singular vectors in the antichiral NS Verma modules  $V_{NS}^a(\mathbf{h})$  and  $V_{NS}^a(-\mathbf{h} - \mathbf{c}/3)$ , on the left-hand side, and in the chiral NS Verma modules  $V_{NS}^{ch}(-\mathbf{h})$  and  $V_{NS}^{ch}(\mathbf{h} + \mathbf{c}/3)$ , on the right-hand side.  $\mathcal{A}_1$  and  $\mathcal{A}_{-1}$  denote the untwisted spectral flow  $\mathcal{A}_\theta$  [22], which will be given in eq. (4.2), for the values  $\theta = \pm 1$ .

### 3 Spectrum of Topological and NS Singular Vectors in Chiral Verma Modules. Subsingular Vectors

We have just shown that the spectrum of  $U(1)$  charges corresponding to the chiral Verma modules which contain topological and NS singular vectors is the same for the singular vectors of the types  $|\chi_T\rangle_l^{(0)G}$ , and  $|\chi_{NS}\rangle_l^{(0)a}$ , on the one hand, and also the same for the singular vectors of the types  $|\chi_T\rangle_l^{(1)G}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ , on the other hand, where the superscript  $a$  indicates that the NS vectors are built on antichiral primaries. Let us introduce  $\mathbf{h}^{(0)}$  and  $\mathbf{h}^{(1)}$  such that the first spectrum consists of all possible values of  $\mathbf{h}^{(0)}$  whereas the second spectrum consists of all possible values of  $\mathbf{h}^{(1)}$ , these values being

connected to each other by the relation  $\mathbf{h}^{(1)} = -\mathbf{h}^{(0)} - \frac{\mathbf{c}}{3}$ . The spectrum of U(1) charges for the case of NS Verma modules built on chiral primaries, in turn, is the corresponding to  $(-\mathbf{h}^{(0)})$ , for singular vectors of the type  $|\chi_{NS}\rangle_l^{(0)ch}$ , and to  $(-\mathbf{h}^{(1)})$ , for singular vectors of the type  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ .

In what follows we will derive a conjecture for the spectrum of possible values of  $\mathbf{h}^{(0)}$  and  $\mathbf{h}^{(1)}$ . These values can be viewed as the roots of the chiral determinants for the Topological algebra, as well as the roots of the antichiral determinants for the NS algebra, whereas  $(-\mathbf{h}^{(0)})$  and  $(-\mathbf{h}^{(1)})$  are the roots of the chiral determinants for the NS algebra. In the case that the values of  $\mathbf{h}^{(0)}$  and/or  $\mathbf{h}^{(1)}$  predict singular vectors which do not exist in the complete Verma modules, *i.e.* only in the chiral Verma modules, then one has encountered subsingular vectors.

### 3.1 Spectrum of Singular Vectors

Let us concentrate on the NS algebra for convenience. In principle we cannot compute the spectra  $\mathbf{h}^{(1)}$  and  $\mathbf{h}^{(0)}$ , corresponding to charged and uncharged NS singular vectors built on antichiral primaries, simply by imposing the relation  $\Delta = -\frac{\mathbf{h}}{2}$  in the spectra given by the roots of the determinant formula for the NS algebra [12],[13],[14]. The reason is that the NS determinant formula does not apply to incomplete Verma modules constructed on chiral or antichiral primary states annihilated by  $G_{-\frac{1}{2}}^+$  or  $G_{-\frac{1}{2}}^-$  (for which, as a result,  $\Delta = \frac{\mathbf{h}}{2}$  or  $\Delta = -\frac{\mathbf{h}}{2}$  respectively, but not the other way around). Nevertheless let us start our analysis with the following ansatz.

*Ansatz.*— The sets of roots of the chiral and antichiral NS determinants coincide with the sets of roots of the NS determinants for the particular cases  $\Delta = \pm\frac{\mathbf{h}}{2}$ , respectively.

Our strategy will be now to analyze the set of roots of the NS determinant formula for  $\Delta = \pm\frac{\mathbf{h}}{2}$  taking into account that, as we deduced in the last section, the charged and uncharged NS singular vectors built on chiral primaries come in pairs, with a precise relation between their corresponding Verma modules. Namely, for singular vectors in antichiral Verma modules, satisfying  $\Delta = -\mathbf{h}/2$ , the relation is  $\mathbf{h}^{(1)} = -\mathbf{h}^{(0)} - \frac{\mathbf{c}}{3}$ , while for singular vectors in chiral Verma modules, satisfying  $\Delta = \mathbf{h}/2$ , it is  $\mathbf{h}^{(1)} = -\mathbf{h}^{(0)} + \frac{\mathbf{c}}{3}$  (in this last expression  $\mathbf{h}^{(1)}$  and  $\mathbf{h}^{(0)}$  denote the true spectra for the chiral Verma modules, instead of  $(-\mathbf{h}^{(1)})$  and  $(-\mathbf{h}^{(0)})$  in our previous notation).

The roots of the determinant formula for the NS algebra [12],[13],[14] are given by the solutions of the quadratic vanishing surface  $f_{rs}^A = 0$ , with

$$f_{r,s}^A = 2 \left( \frac{\mathbf{c}-3}{3} \right) \Delta - \mathbf{h}^2 - \frac{1}{4} \left( \frac{\mathbf{c}-3}{3} \right)^2 + \frac{1}{4} \left( \left( \frac{\mathbf{c}-3}{3} \right) r + s \right)^2 \quad r \in \mathbf{Z}^+, s \in 2\mathbf{Z}^+ \quad (3.1)$$

and the solutions of the vanishing plane  $g_k^A = 0$ , with

$$g_k^A = 2\Delta - 2kh + \left(\frac{c-3}{3}\right)(k^2 - \frac{1}{4}) \quad k \in \mathbf{Z} + \frac{1}{2} \quad (3.2)$$

In the complete Verma modules the solutions to  $f_{rs}^A = 0$  and  $g_k^A = 0$  correspond to uncharged and charged singular vectors, respectively.

Let us concentrate on the antichiral case, for convenience. Solving for  $f_{r,s}^A = 0$ , with  $\Delta = -\frac{h}{2}$ , one finds two two-parameter solutions for  $h$  (since  $f_{r,s}^A = 0$  becomes a quadratic equation for  $h$ ). These solutions are

$$h_{r,s} = -\frac{1}{2} \left( \left( \frac{c-3}{3} \right) (r+1) + s \right) \quad (3.3)$$

and

$$\hat{h}_{r,s} = \frac{1}{2} \left( \left( \frac{c-3}{3} \right) (r-1) + s \right). \quad (3.4)$$

Solving for  $g_k^A = 0$ , with  $\Delta = -\frac{h}{2}$ , one finds the one-parameter solution

$$h_k = \left( \frac{c-3}{6} \right) \left( k - \frac{1}{2} \right), \quad (3.5)$$

except for  $k = -\frac{1}{2}$  where  $g_k^A$  is identically zero (*i.e.* all the states  $G_{1/2}^-|\Delta, h\rangle$  with  $\Delta = -\frac{h}{2}$  are singular vectors).

If our ansatz is correct, the set of roots of the antichiral NS determinant formula should be equal as the set of roots given by the solutions (3.3), (3.4) and (3.5). The problem at hand is therefore to distribute all these solutions into two sets, say  $H^{(0)}$  and  $H^{(1)}$ , such that for any given solution  $h^{(0)}$  in the set  $H^{(0)}$  there exists one solution  $h^{(1)}$  in the set  $H^{(1)}$ , satisfying  $h^{(0)} = -h^{(1)} - \frac{c}{3}$ , and vice-versa. For this purpose it is helpful to write down the expressions corresponding to  $(-h_{r,s} - \frac{c}{3})$ ,  $(-\hat{h}_{r,s} - \frac{c}{3})$  and  $(-h_k - \frac{c}{3})$ . These are given by

$$-h_{r,s} - \frac{c}{3} = \frac{1}{2} \left( \left( \frac{c-3}{3} \right) (r-1) + s - 2 \right), \quad (3.6)$$

$$-\hat{h}_{r,s} - \frac{c}{3} = -\frac{1}{2} \left( \left( \frac{c-3}{3} \right) (r+1) + s + 2 \right) \quad (3.7)$$

and

$$-h_k - \frac{c}{3} = -\frac{1}{2} \left( \left( \frac{c-3}{3} \right) \left( k + \frac{3}{2} \right) + 2 \right). \quad (3.8)$$

Comparing these expressions with the set of solutions  $\mathbf{h}_{r,s}$ ,  $\hat{\mathbf{h}}_{r,s}$  and  $\mathbf{h}_k$ , given by (3.3), (3.4) and (3.5), one finds straightforwardly

$$\mathbf{h}_{r,2} = -\mathbf{h}_{r-\frac{1}{2}} - \frac{\mathbf{c}}{3}, \quad \mathbf{h}_{r,s>2} = -\hat{\mathbf{h}}_{r,(s-2)} - \frac{\mathbf{c}}{3}. \quad (3.9)$$

Therefore the simplest solution to the problem is that the spectrum corresponding to the uncharged singular vectors  $|\chi_{NS}\rangle_l^{(0)a}$  is given by  $\mathbf{h}_{r,s}^{(0)} = \mathbf{h}_{r,s}$ , eq. (3.3), with the level of the state  $l = \frac{rs}{2}$  as usual [12], whereas the spectrum corresponding to the charged singular vectors  $|\chi_{NS}\rangle_{l-1/2}^{(1)a}$ , at level  $l - \frac{1}{2} = \frac{rs-1}{2}$ , is given by  $\mathbf{h}_{r,s}^{(1)} = -\mathbf{h}_{r,s} - \frac{\mathbf{c}}{3}$ , that is

$$\mathbf{h}_{r,s}^{(0)} = -\frac{1}{2} \left( \left( \frac{\mathbf{c}-3}{3} \right) (r+1) + s \right) \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+, \quad (3.10)$$

and

$$\mathbf{h}_{r,s}^{(1)} = \frac{1}{2} \left( \left( \frac{\mathbf{c}-3}{3} \right) (r-1) + s - 2 \right) \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+, \quad (3.11)$$

with  $\mathbf{h}_{r,s}^{(1)}$  containing the two series  $\mathbf{h}_k$  and  $\hat{\mathbf{h}}_{r,s}$ . Namely, for  $s = 2$   $\mathbf{h}_{r,2}^{(1)} = \mathbf{h}_{r-\frac{1}{2}}$ , where  $l - \frac{1}{2} = r - \frac{1}{2}$  is the level of the charged singular vector, while for  $s > 2$   $\mathbf{h}_{r,s>2}^{(1)} = \hat{\mathbf{h}}_{r,(s-2)}$ , with the level given by  $l - \frac{1}{2} = \frac{rs-1}{2}$ . This implies the existence of *charged subsingular* vectors in the complete Verma modules with  $\Delta = -\mathbf{h}/2$ ,  $\mathbf{h} = \mathbf{h}_{r,s>2}^{(1)}$ , since these Verma modules contain uncharged singular vectors rather than charged. The complete Verma modules with  $\Delta = -\mathbf{h}/2$ ,  $\mathbf{h} = \mathbf{h}_{r,2}^{(1)}$ , however, contain charged singular vectors like the antichiral Verma modules.

We see therefore that for this solution half of the zeroes of the quadratic vanishing surface  $f_{r,s}^A = 0$ , with  $\Delta = -\mathbf{h}/2$ , for every pair  $(r, s)$ , correspond to uncharged singular vectors  $|\chi_{NS}\rangle_l^{(0)a}$  at level  $l = \frac{rs}{2}$ , and the other half correspond to charged singular vectors  $|\chi_{NS}\rangle_{l-1/2}^{(1)a}$ , which are subsingular in the complete Verma modules, at level  $l - \frac{1}{2} = \frac{r(s+2)-1}{2}$ . The zeroes of the vanishing plane  $g_k^A = 0$ , for  $\Delta = -\mathbf{h}/2$ , in turn, correspond to charged singular vectors  $|\chi_{NS}\rangle_{l-1/2}^{(1)a}$  at level  $l - \frac{1}{2} = k$ , with  $k > 0$ .

Before searching for more intricate solutions let us have a look at the data for  $\mathbf{h}^{(0)}$  and  $\mathbf{h}^{(1)}$  given by the singular vectors themselves. We have computed until level 3 all the topological singular vectors in chiral topological Verma modules and all the NS singular vectors in chiral and antichiral NS Verma modules. The explicit expressions for the topological singular vectors we have given recently in [11] (some of those singular vectors were already published). The explicit expressions for the NS singular vectors will be given in [26], although in the Appendix we also write down and analyze the NS singular vectors  $|\chi_{NS}\rangle_1^{(0)}$ ,  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$  and  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$  in chiral, antichiral and complete Verma modules. Here we need only the values of  $\mathbf{h}^{(0)}$  and  $\mathbf{h}^{(1)}$ . These are the following:

- For  $|\chi_{NS}\rangle_1^{(0)a}$ ,  $|\chi_T\rangle_1^{(0)G}$  and  $|\chi_T\rangle_1^{(-1)Q}$   $\mathbf{h}^{(0)} = -\frac{\mathbf{c}}{3}$
- For  $|\chi_{NS}\rangle_{\frac{1}{2}}^{(1)a}$ ,  $|\chi_T\rangle_1^{(1)G}$  and  $|\chi_T\rangle_1^{(0)Q}$   $\mathbf{h}^{(1)} = 0$
- For  $|\chi_{NS}\rangle_2^{(0)a}$ ,  $|\chi_T\rangle_2^{(0)G}$  and  $|\chi_T\rangle_2^{(-1)Q}$   $\mathbf{h}^{(0)} = \frac{1-\mathbf{c}}{2}$  ,  $-\frac{\mathbf{c}+3}{3}$
- For  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ ,  $|\chi_T\rangle_2^{(1)G}$  and  $|\chi_T\rangle_2^{(0)Q}$   $\mathbf{h}^{(1)} = \frac{\mathbf{c}-3}{6}$  ,  $1$
- For  $|\chi_{NS}\rangle_3^{(0)a}$ ,  $|\chi_T\rangle_3^{(0)G}$  and  $|\chi_T\rangle_3^{(-1)Q}$   $\mathbf{h}^{(0)} = \frac{3-2\mathbf{c}}{3}$  ,  $-\frac{\mathbf{c}+6}{3}$
- For  $|\chi_{NS}\rangle_{\frac{5}{2}}^{(1)a}$ ,  $|\chi_T\rangle_3^{(1)G}$  and  $|\chi_T\rangle_3^{(0)Q}$   $\mathbf{h}^{(1)} = \frac{\mathbf{c}-3}{3}$  ,  $2$

(we remind that the topological singular vectors of types  $|\chi_T\rangle^{(1)G}$  and  $|\chi_T\rangle^{(0)Q}$  are together in the same Verma module, at the same level). In addition, the BRST-invariant uncharged topological singular vector at level 4, *i.e.*  $|\chi_T\rangle_4^{(0)Q}$ , has been computed in [8] with the result  $\mathbf{h}^{(1)} = \frac{\mathbf{c}-3}{2}$ ,  $\frac{\mathbf{c}+3}{6}$ , 3. These values also correspond to the singular vectors  $|\chi_T\rangle_4^{(1)G}$  and  $|\chi_{NS}\rangle_{\frac{7}{2}}^{(1)a}$ , as we have deduced.

By comparing these results with the roots of the NS determinant formula for  $\Delta = -\mathbf{h}/2$ , given by expressions (3.3), (3.4) and (3.5), we notice that the values we have found for  $\mathbf{h}^{(0)}$  fit exactly in the expression  $\mathbf{h}_{r,s}$  (3.3), that is, the upper solution for the quadratic vanishing surface, but not in the lower solution  $\hat{\mathbf{h}}_{r,s}$ . The values we have found for  $\mathbf{h}^{(1)}$  follow exactly the prediction of the spectral flow automorphism for each case, *i.e.*  $\mathbf{h}^{(1)} = -\mathbf{h}^{(0)} - \frac{\mathbf{c}}{3}$ . Hence the actual spectrum of U(1) charges  $\mathbf{h}^{(0)}$  and  $\mathbf{h}^{(1)}$ , as far as we can tell, follows the pattern we have found under the ansatz that the set of roots of the antichiral NS determinant formula coincide with the set of roots of the NS determinant formula for  $\Delta = -\frac{\mathbf{h}}{2}$ . That is,  $\mathbf{h}_{r,s}^{(0)} = \mathbf{h}_{r,s}$  is given by (3.10) and  $\mathbf{h}_{r,s}^{(1)} = -\mathbf{h}_{r,s} - \frac{\mathbf{c}}{3}$  is given by (3.11)<sup>‡</sup>.

For the case of NS singular vectors in chiral Verma modules one finds the same values for the U(1) charges as for the case of antichiral Verma modules but with the sign reversed, as expected. Therefore the spectra of U(1) charges for the NS singular vectors of the types  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$  are given by  $(-\mathbf{h}_{r,s}^{(0)})$  and  $(-\mathbf{h}_{r,s}^{(1)})$  respectively.

---

<sup>‡</sup>The spectrum  $\mathbf{h}_{r,s}^{(1)}$ , eq. (3.11), was written for the first time in ref. [8], eq. (3.1), (with  $s \in \mathbf{Z}^+$ ) fitting the data obtained from the computation of the topological singular vectors of type  $|\chi_T\rangle^{(0)Q}$ , referred simply as “the topological singular vectors”. However this spectrum was never related to charged singular vectors of the NS algebra, the author believed that it was related to uncharged NS singular vectors instead, and was given directly by the uncharged roots of the NS determinant formula for the case  $\Delta = -\frac{\mathbf{h}}{2}$ . The fact that this spectrum corresponds to *charged singular* vectors in chiral NS Verma modules is precisely the reason why it corresponds, for the values  $\mathbf{h}_{r,s>2}^{(1)}$ , to *charged subsingular* vectors in the complete Verma modules.

*Summary of Results.*— Let us summarize our results for the spectra of  $U(1)$  charges and conformal weights corresponding to the Verma modules which contain the different kinds of singular vectors. Although these results have been checked only until level 4 we conjecture that they hold at any level. We remind that  $\mathbf{h}_{r,s}^{(0)}$  is given by (3.10) and  $\mathbf{h}_{r,s}^{(1)} = -\mathbf{h}_{r,s}^{(0)} - \frac{c}{3}$  is given by (3.11).

1) The spectrum of  $U(1)$  charges corresponding to the chiral topological Verma modules which contain singular vectors of the types  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(-1)Q}$ , on the one hand, and singular vectors of the types  $|\chi_T\rangle_l^{(1)G}$  and  $|\chi_T\rangle_l^{(0)Q}$ , on the other hand, is given by  $\mathbf{h}_{r,s}^{(0)}$ , and  $\mathbf{h}_{r,s}^{(1)}$ , respectively, with  $l = \frac{rs}{2}$ . These are the only types of topological singular vectors which exist in chiral topological Verma modules.

2) The spectra of  $U(1)$  charges and conformal weights corresponding to antichiral NS Verma modules which contain uncharged and charged NS singular vectors,  $|\chi_{NS}\rangle_l^{(0)a}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ , are given by  $\mathbf{h}_{r,s}^{(0)}$ ,  $\Delta_{r,s}^{(0)} = -\frac{1}{2}\mathbf{h}_{r,s}^{(0)}$ , and  $\mathbf{h}_{r,s}^{(1)}$ ,  $\Delta_{r,s}^{(1)} = -\frac{1}{2}\mathbf{h}_{r,s}^{(1)}$ , respectively, with  $l = \frac{rs}{2}$ . These are the only types of NS singular vectors which exist in antichiral NS Verma modules.

3) The spectra of  $U(1)$  charges and conformal weights corresponding to chiral NS Verma modules which contain uncharged and charged NS singular vectors,  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ , are given by  $(-\mathbf{h}_{r,s}^{(0)})$ ,  $\Delta_{r,s}^{(0)} = \frac{1}{2}(-\mathbf{h}_{r,s}^{(0)})$ , and  $(-\mathbf{h}_{r,s}^{(1)})$ ,  $\Delta_{r,s}^{(1)} = \frac{1}{2}(-\mathbf{h}_{r,s}^{(1)})$ , respectively, with  $l = \frac{rs}{2}$ . These are the only types of NS singular vectors which exist in chiral NS Verma modules. Therefore the spectrum of conformal weights is the same for chiral NS Verma modules as for antichiral NS Verma modules, although the spectrum of  $U(1)$  charges, as well as the relative charge of the singular vectors, is reversed in sign.

## 3.2 Subsingular Vectors

Subsingular vectors are singular vectors only in the quotient of complete Verma modules by submodules generated by singular vectors. This is indeed the case at hand because chiral and antichiral NS Verma modules are nothing but the quotient of complete NS Verma modules by the singular vectors  $G_{-1/2}^+|\Delta, \mathbf{h}\rangle$ ,  $\Delta = \mathbf{h}/2$ , and  $G_{-1/2}^-|\Delta, \mathbf{h}\rangle$ ,  $\Delta = -\mathbf{h}/2$ , respectively. Similarly, the chiral topological Verma modules are the quotient of complete topological Verma modules of types  $V_T(|0, \mathbf{h}\rangle^Q)$  and  $V_T(|0, \mathbf{h}\rangle^G)$ , by the submodules generated by the singular vectors  $\mathcal{G}_0|0, \mathbf{h}\rangle$  and  $\mathcal{Q}_0|0, \mathbf{h}\rangle$ , respectively (in ref. [11] there is a detailed description of the complete topological Verma modules and the spectra of conformal weights and  $U(1)$  charges corresponding to singular vectors).

We have found the remarkable fact that (at least until level 4) half of the uncharged

NS singular vectors with levels  $l = \frac{rs}{2}$  in the complete Verma modules with  $\Delta = \mp \frac{\mathbf{h}}{2}$ ,  $\mathbf{h} = \pm \hat{\mathbf{h}}_{r,s}$ , eq. (3.4), or equivalently  $\mathbf{h} = \pm \mathbf{h}_{r,(s+2)}^{(1)}$ , eq. (3.11), have been replaced by charged NS singular vectors with levels  $l - \frac{1}{2} = \frac{r(s+2)-1}{2}$ , in the antichiral and chiral Verma modules, respectively. These charged singular vectors are therefore charged subsingular vectors in the complete Verma modules. Since  $s \in 2\mathbf{Z}^+$ , the complete Verma modules with  $\Delta = \mp \frac{\mathbf{h}}{2}$  which contain charged subsingular vectors are those with  $\mathbf{h} = \pm \mathbf{h}_{r,s>2}^{(1)}$  (in other words, the set of U(1) charges given by  $\mathbf{h}_{r,s>2}^{(1)}$  is equal to the set given by  $\hat{\mathbf{h}}_{r,s}$ , as we deduced in the previous subsection), whereas the complete Verma modules with  $\mathbf{h} = \pm \mathbf{h}_{r,2}^{(1)}$  do not contain charged subsingular vectors, but charged singular vectors instead (which are singular both in the complete Verma modules and in the chiral Verma modules).

In the case of the Topological algebra there is a much more symmetrical situation with respect to charged and uncharged singular and subsingular vectors. Namely, there are charged as well as uncharged subsingular vectors, and charged as well as uncharged singular vectors which vanish in the chiral Verma modules. One finds<sup>§</sup> that  $\mathbf{h} = \mathbf{h}_{r,s>2}^{(1)}$  corresponds to charged and uncharged subsingular vectors of types  $|\chi_T\rangle_l^{(1)G}$  and  $|\chi_T\rangle_l^{(0)Q}$ , in the complete Verma modules  $V_T(|0, \mathbf{h}\rangle^G)$ , whereas  $\mathbf{h} = \mathbf{h}_{r,s>2}^{(0)}$  corresponds to charged and uncharged subsingular vectors of types  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(-1)Q}$  in the complete Verma modules  $V_T(|0, \mathbf{h}\rangle^Q)$ . All of these topological subsingular vectors become singular in the chiral topological Verma modules  $V_T(|0, \mathbf{h}\rangle^{G,Q})$ .

Under the untwistings  $T_{W1}$  and  $T_{W2}$  only the  $\mathcal{G}_0$ -closed topological h.w. states (primary and singular vectors) remain h.w. states of the NS algebra. Therefore the complete topological Verma modules of type  $V(|0, \mathbf{h}\rangle^Q)$  are not transformed into NS Verma modules, and the BRST-invariant uncharged topological singular vectors  $|\chi_T\rangle_l^{(0)Q}$ , in the chiral topological Verma modules, are not transformed into singular vectors in the chiral or antichiral NS Verma modules. In this manner the symmetry between charged and uncharged topological subsingular vectors is broken under the untwistings, and one only finds charged subsingular vectors in the NS algebra, for Verma modules with  $\Delta = \mp \frac{\mathbf{h}}{2}$ ,  $\mathbf{h} = \mp \mathbf{h}_{r,s>2}^{(1)}$ .

In the Appendix we analyze the complete situation for the particular case of the uncharged singular vectors  $|\chi_{NS}\rangle_1^{(0)}$  and the charged singular vectors  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$  and  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$ . We write down the h.w. equations, with their solutions, for the primary states being non-chiral, chiral and antichiral. We show that the uncharged singular vectors  $|\chi_{NS}\rangle_1^{(0)}$ , for  $\Delta = \mp \frac{\mathbf{h}}{2}$ ,  $\mathbf{h} = \pm \hat{\mathbf{h}}_{1,2} = \pm 1$ , or equivalently  $\mathbf{h} = \pm \mathbf{h}_{1,4}^{(1)} = \pm 1$ , vanish once we switch on antichirality and chirality on the primary states, respectively, whereas the charged subsingular vectors become the charged singular vectors  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$  with  $\mathbf{h} = 1$ , and  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$ .

---

<sup>§</sup>The subsingular vectors of the Topological algebra were not considered in the first version of this paper. Here we borrow our own results from ref. [11], where we have published all the topological subsingular vectors at levels 2 and 3 which become singular in chiral topological Verma modules.



with  $\mathbf{h} = -1$ . We also show that the subsingular vectors are non-highest weight null vectors, after we switch off antichirality and chirality respectively on the primary states, which are not descendant states of any singular vectors, as expected (they can descend down to a singular vector but not the other way around).

A final remark is that subsingular vectors do not exist for Verma modules of the Virasoro algebra neither for Verma modules of the  $\mathfrak{sl}(2)$  algebra. The very existence of these objects for the  $N=2$  Superconformal algebras has been so far unknown<sup>¶</sup>, in spite of the fact that the issue has been investigated during the last few years [27]. Observe that the subsingular vectors that we have found are those which become singular in chiral Verma modules. The issue whether or not these are the only subsingular vectors of the Topological and NS algebras is currently under investigation.

## 4 Spectrum of R Singular Vectors on Ramond Ground States. Subsingular Vectors

The singular vectors of the NS algebra transform into singular vectors of the R algebra under the action of the spectral flows, and vice-versa [19],[20],[21],[22]. In particular, the NS singular vectors built on chiral or antichiral primaries transform into R singular vectors built on the Ramond ground states. As a consequence, we can write down the spectrum of  $U(1)$  charges corresponding to the R Verma modules built on the R ground states which contain singular vectors, simply by applying the spectral flow transformations to the spectra (3.10) and (3.11) found in last section. Before doing this let us say a few words about the R algebra.

*The R Algebra.*— The Periodic  $N=2$  Superconformal algebra is given by (2.7), where the fermionic generators  $G_r^\pm$  are integer moded. The zero modes of the fermionic generators characterize the states as being  $G_0^+$ -closed or  $G_0^-$ -closed, as the anticommutator  $\{G_0^+, G_0^-\} = 2L_0 - \frac{c}{12}$  shows. The R ground states are annihilated by both  $G_0^+$  and  $G_0^-$ , therefore the conformal weight satisfies  $\Delta = \frac{c}{24}$  for them, as a result.

In order to simplify the analysis that follows it is very convenient to define the  $U(1)$  charge for the states of the R algebra in the same way as for the states of the NS algebra. Namely, the  $U(1)$  charge of the states will be denoted by  $\mathbf{h}$ , instead of  $\mathbf{h} \pm \frac{1}{2}$ , whereas the relative charge  $q$  of a secondary state will be defined as the difference between the  $H_0$ -eigenvalue of the state and the  $H_0$ -eigenvalue of the primary on which it is built. Therefore, the relative charges of the R states are defined to be integer, in contrast with

---

<sup>¶</sup>unknown until the first version of this paper appeared in hep-th/9602166 (1996)

the usual definition. We will denote the R singular vectors as  $|\chi_R\rangle_l^{(q)+}$  and  $|\chi_R\rangle_l^{(q)-}$ , where, in addition to the level and the relative charge, we indicate that the vector is annihilated by  $G_0^+$  or  $G_0^-$ .

*The Spectral Flows.* – The “usual” spectral flow [19], [20], is given by the one-parameter family of transformations

$$\begin{aligned}\mathcal{U}_\theta L_m \mathcal{U}_\theta^{-1} &= L_m + \theta H_m + \frac{\epsilon}{6}\theta^2 \delta_{m,0}, \\ \mathcal{U}_\theta H_m \mathcal{U}_\theta^{-1} &= H_m + \frac{\epsilon}{3}\theta \delta_{m,0}, \\ \mathcal{U}_\theta G_r^+ \mathcal{U}_\theta^{-1} &= G_{r+\theta}^+, \\ \mathcal{U}_\theta G_r^- \mathcal{U}_\theta^{-1} &= G_{r-\theta}^-, \end{aligned} \tag{4.1}$$

satisfying  $\mathcal{U}_\theta^{-1} = \mathcal{U}_{(-\theta)}$  and giving rise to isomorphic algebras. If we denote by  $(\Delta, \mathbf{h})$  the  $(L_0, H_0)$  eigenvalues of any given state, then the eigenvalues of the transformed state  $\mathcal{U}_\theta|\chi\rangle$  are  $(\Delta - \theta\mathbf{h} + \frac{\epsilon}{6}\theta^2, \mathbf{h} - \frac{\epsilon}{3}\theta)$ . If the state  $|\chi\rangle$  is a level- $l$  secondary state with relative charge  $q$ , then one gets straightforwardly that the level of the transformed state  $\mathcal{U}_\theta|\chi\rangle$  changes to  $l - \theta q$ , while the relative charge remains equal.

There is another spectral flow [22], [23], mirrored to the previous one, which is the untwisting of the Topological algebra automorphism (2.4) (for general values of  $\theta$ ), given by

$$\begin{aligned}\mathcal{A}_\theta L_m \mathcal{A}_\theta &= L_m + \theta H_m + \frac{\epsilon}{6}\theta^2 \delta_{m,0}, \\ \mathcal{A}_\theta H_m \mathcal{A}_\theta &= -H_m - \frac{\epsilon}{3}\theta \delta_{m,0}, \\ \mathcal{A}_\theta G_r^+ \mathcal{A}_\theta &= G_{r-\theta}^-, \\ \mathcal{A}_\theta G_r^- \mathcal{A}_\theta &= G_{r+\theta}^+. \end{aligned} \tag{4.2}$$

with  $\mathcal{A}_\theta^{-1} = \mathcal{A}_\theta$ . The  $(L_0, H_0)$  eigenvalues of the transformed states  $\mathcal{A}_\theta|\chi\rangle$  are now  $(\Delta + \theta\mathbf{h} + \frac{\epsilon}{6}\theta^2, -\mathbf{h} - \frac{\epsilon}{3}\theta)$  (that is, they differ from the previous case by the interchange  $\mathbf{h} \rightarrow -\mathbf{h}$ ). From this one easily deduces that, under the spectral flow  $\mathcal{A}_\theta$ , the level  $l$  of any secondary state changes to  $l + \theta q$  while the relative charge  $q$  reverses its sign.

For half-integer values of  $\theta$  the two spectral flows interpolate between the NS algebra and the R algebra. In particular, for  $\theta = \frac{1}{2}$  the h.w. states of the NS algebra (primaries as well as singular vectors) become h.w. states of the R algebra with helicity  $(-)$  (*i.e.* annihilated by  $G_0^-$ ), while for  $\theta = -\frac{1}{2}$  the h.w. states of the NS algebra become h.w. states of the R algebra with helicity  $(+)$  (*i.e.* annihilated by  $G_0^+$ ). In addition,  $\mathcal{U}_{1/2}$  and  $\mathcal{A}_{-1/2}$  map the chiral primaries of the NS algebra (*i.e.* annihilated by  $G_{-1/2}^+$ ) into the set of R ground states, whereas  $\mathcal{U}_{-1/2}$  and  $\mathcal{A}_{1/2}$  map the antichiral primaries (*i.e.* annihilated by  $G_{-1/2}^-$ ) into the set of R ground states. As a result, the spectral flows (4.1) and (4.2) transform the NS singular vectors built on chiral and antichiral primaries into R singular vectors built on the R ground states.

## 4.1 Spectrum of Singular Vectors

We showed in section 2 that the NS singular vectors built on chiral primaries are only of two types,  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$ , and similarly, the NS singular vectors built on antichiral primaries come only in two types  $|\chi_{NS}\rangle_l^{(0)a}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ . In fact,  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_l^{(0)a}$ , on the one hand, and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ , on the other hand, are mirrored under the interchange  $H_m \leftrightarrow -H_m$ ,  $G_r^+ \leftrightarrow G_r^-$ , because they are the two possible untwistings of the  $\mathcal{G}_0$ -closed topological singular vectors  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(1)G}$  respectively.

From the results just discussed one deduces straightforwardly that the NS singular vectors  $|\chi_{NS}\rangle_l^{(0)a}$ ,  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a}$ ,  $|\chi_{NS}\rangle_l^{(0)ch}$  and  $|\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch}$  are transformed, by the spectral flows (4.1) and (4.2), into R singular vectors built on R ground states, in the following way:

$$\mathcal{A}_{1/2} |\chi_{NS}\rangle_l^{(0)a} = \mathcal{U}_{1/2} |\chi_{NS}\rangle_l^{(0)ch} = |\chi_R\rangle_l^{(0)-} \quad (4.3)$$

$$\mathcal{A}_{1/2} |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a} = \mathcal{U}_{1/2} |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch} = |\chi_R\rangle_l^{(-1)-} \quad (4.4)$$

$$\mathcal{U}_{-1/2} |\chi_{NS}\rangle_l^{(0)a} = \mathcal{A}_{-1/2} |\chi_{NS}\rangle_l^{(0)ch} = |\chi_R\rangle_l^{(0)+} \quad (4.5)$$

$$\mathcal{U}_{-1/2} |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(1)a} = \mathcal{A}_{-1/2} |\chi_{NS}\rangle_{l-\frac{1}{2}}^{(-1)ch} = |\chi_R\rangle_l^{(1)+} \quad (4.6)$$

with  $|\chi_R\rangle_l^{(0)+}$  and  $|\chi_R\rangle_l^{(0)-}$  on the one hand, and  $|\chi_R\rangle_l^{(1)+}$  and  $|\chi_R\rangle_l^{(-1)-}$  on the other hand, mirrored to each other. Observe that the charged singular vectors built on the R ground states come only in two types:  $|\chi_R\rangle_l^{(1)+}$  and  $|\chi_R\rangle_l^{(-1)-}$ , *i.e.* the sign of the relative charge is equal to the sign of the “helicity”. The chiral and antichiral NS Verma modules  $V_{NS}^{ch}(\mathbf{h})$  and  $V_{NS}^a(\mathbf{h})$  are transformed, in turn, in R Verma modules built on R ground states as

$$\begin{aligned} \mathcal{U}_{1/2} V_{NS}^{ch}(\mathbf{h}) &\rightarrow V_R(\mathbf{h} - \frac{\mathbf{c}}{6}) , & \mathcal{U}_{-1/2} V_{NS}^a(\mathbf{h}) &\rightarrow V_R(\mathbf{h} + \frac{\mathbf{c}}{6}) , \\ \mathcal{A}_{1/2} V_{NS}^a(\mathbf{h}) &\rightarrow V_R(-\mathbf{h} - \frac{\mathbf{c}}{6}) , & \mathcal{A}_{-1/2} V_{NS}^{ch}(\mathbf{h}) &\rightarrow V_R(-\mathbf{h} + \frac{\mathbf{c}}{6}) . \end{aligned}$$

Hence, whereas the spectrum of conformal weights for all the R Verma modules built on R ground states is just given by  $\Delta = \frac{\mathbf{c}}{24}$ , the spectrum of U(1) charges for those which contain singular vectors is given by:  $\mathbf{h}_{r,s}^{(0)+} = \mathbf{h}_{r,s}^{(0)} + \frac{\mathbf{c}}{6}$  for singular vectors of type  $|\chi_R\rangle_l^{(0)+}$ ,

$\mathbf{h}_{r,s}^{(1)+} = \mathbf{h}_{r,s}^{(1)} + \frac{\mathbf{c}}{6}$  for singular vectors of type  $|\chi_R\rangle_l^{(1)+}$ ,  $\mathbf{h}_{r,s}^{(0)-} = -(\mathbf{h}_{r,s}^{(0)} + \frac{\mathbf{c}}{6})$  for singular vectors of type  $|\chi_R\rangle_l^{(0)-}$  and  $\mathbf{h}_{r,s}^{(-1)-} = -(\mathbf{h}_{r,s}^{(1)} + \frac{\mathbf{c}}{6})$  for singular vectors of type  $|\chi_R\rangle_l^{(-1)-}$ . Using the expressions for  $\mathbf{h}_{r,s}^{(0)}$  and  $\mathbf{h}_{r,s}^{(1)}$  given by (3.10) and (3.11) one obtains:

$$\mathbf{h}_{r,s}^{(0)+} = \mathbf{h}_{r,s}^{(-1)-} = -\frac{1}{2} \left( \left( \frac{\mathbf{c}-3}{3} \right) r + s - 1 \right) \text{ for } |\chi_R\rangle_l^{(0)+} \text{ and } |\chi_R\rangle_l^{(-1)-}, \quad (4.7)$$

$$\mathbf{h}_{r,s}^{(0)-} = \mathbf{h}_{r,s}^{(1)+} = \frac{1}{2} \left( \left( \frac{\mathbf{c}-3}{3} \right) r + s - 1 \right) \text{ for } |\chi_R\rangle_l^{(0)-} \text{ and } |\chi_R\rangle_l^{(1)+}. \quad (4.8)$$

We see that  $\mathbf{h}_{r,s}^{(0)+} = \mathbf{h}_{r,s}^{(-1)-} = -\mathbf{h}_{r,s}^{(0)-} = -\mathbf{h}_{r,s}^{(1)+}$ . Therefore the singular vectors of the types  $|\chi_R\rangle_l^{(0)+}$  and  $|\chi_R\rangle_l^{(-1)-}$  are together in the same Verma module  $V_R(\mathbf{h})$  and at the same level, and so are the mirrored singular vectors of the types  $|\chi_R\rangle_l^{(0)-}$  and  $|\chi_R\rangle_l^{(1)+}$ , which belong to the mirror Verma module  $V_R(-\mathbf{h})$ . In other words, the singular vectors built on the R ground states come always in sets of two mirrored pairs at the same level. Every pair consists of one charged and one uncharged vector, with opposite helicities, one pair belonging to the Verma module  $V_R(\mathbf{h})$  and the mirrored pair belonging to the mirror Verma module  $V_R(-\mathbf{h})$ .

This family structure is easy to see also taking into account that the action of  $G_0^+$  or  $G_0^-$  on any R singular vector built on the R ground states produces another singular vector with different relative charge and different helicity but with the same level and sitting in the same Verma module. This resembles very much the family structure for the topological singular vectors built in chiral Verma modules, that we analyzed in section 2. However, there is a drastic difference here because in the latter case the four members of the topological family, *i.e.*  $|\chi_T\rangle_l^{(0)G}$ ,  $|\chi_T\rangle_l^{(0)Q}$ ,  $|\chi_T\rangle_l^{(1)G}$  and  $|\chi_T\rangle_l^{(-1)Q}$ , are completely different from each other, while in this case the four members are two by two mirror symmetric under the interchange  $H_m \leftrightarrow -H_m$ ,  $G_r^+ \leftrightarrow G_r^-$ .

## 4.2 Subsingular Vectors

Now let us compare the spectra we have found, eqns. (4.7) and (4.8), with the roots of the determinant formula for the R algebra [12], [14], specialized to the case  $\Delta = \frac{\mathbf{c}}{24}$ . The roots of the determinant formula for the R algebra are given, in ref. [12], by the vanishing quadratic surface  $f_{r,s}^P = 0$  and the vanishing plane  $g_k^P = 0$ , where

$$f_{r,s}^P = 2 \left( \frac{\mathbf{c}-3}{3} \right) \left( \Delta - \frac{\mathbf{c}}{24} \right) - \mathbf{h}^2 + \frac{1}{4} \left( \left( \frac{\mathbf{c}-3}{3} \right) r + s \right)^2 \quad r \in \mathbf{Z}^+, s \in 2\mathbf{Z}^+ \quad (4.9)$$

and

$$g_k^P = 2\Delta - 2k\mathfrak{h} + \left(\frac{\mathfrak{c}-3}{3}\right) \left(k^2 - \frac{1}{4}\right) - \frac{1}{4} \quad k \in \mathbf{Z} + \frac{1}{2}. \quad (4.10)$$

For  $\Delta = \frac{\mathfrak{c}}{24}$  they result in the following solutions

$$\mathfrak{h}_{r,s}^{BFK} = \pm \frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) r + s \right) \quad (4.11)$$

and

$$\mathfrak{h}_k^{BFK} = \frac{1}{2} \left( \frac{\mathfrak{c}-3}{3} \right) k \quad (4.12)$$

where the superscript BFK indicates that these are U(1) charges in the notation of ref. [12]. Therefore we have to add  $(\pm \frac{1}{2})$  to these expressions to translate them into our notation. Namely,

$$\mathfrak{h}_{r,s}^{BFK} + \frac{1}{2} = \begin{cases} \frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) r + s + 1 \right) \\ -\frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) r + s - 1 \right) \end{cases} \quad (4.13)$$

$$\mathfrak{h}_{r,s}^{BFK} - \frac{1}{2} = \begin{cases} \frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) r + s - 1 \right) \\ -\frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) r + s + 1 \right) \end{cases} \quad (4.14)$$

and

$$\mathfrak{h}_k^{BFK} + \frac{1}{2} = \frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) k + 1 \right) \quad (4.15)$$

$$\mathfrak{h}_k^{BFK} - \frac{1}{2} = \frac{1}{2} \left( \left( \frac{\mathfrak{c}-3}{3} \right) k - 1 \right) \quad (4.16)$$

Comparing these expressions with the spectra given by (4.7) and (4.8), we see that the situation is the same as for the NS algebra for Verma modules built on chiral primaries. Namely, half of the zeroes of  $f_{r,s}^P = 0$ , for every pair  $(r, s)$ , correspond to uncharged singular vectors and the other half correspond to charged singular vectors. However, since now charged and uncharged singular vectors (with different helicities) share the same spectra, this statement is rather ambiguous unless we specify the helicities. Let us make the choice that adding  $\frac{1}{2}$  (or  $-\frac{1}{2}$ ) to the BFK spectra (4.11) and (4.12) one gets the spectra corresponding to the helicity  $+$  (or  $-$ ) singular vectors. Then one finds the following identifications. The upper solution of  $\mathfrak{h}_{r,s}^{BFK} + \frac{1}{2}$  corresponds to the charged singular vectors  $|\chi_R\rangle_l^{(1)+}$  at level  $\frac{r(s+2)}{2}$ , whereas the lower solution corresponds to the uncharged singular vectors  $|\chi_R\rangle_l^{(0)+}$  at level  $\frac{rs}{2}$ . The upper solution of  $\mathfrak{h}_{r,s}^{BFK} - \frac{1}{2}$  corresponds to

the uncharged singular vectors  $|\chi_R\rangle_l^{(0)-}$  at level  $\frac{rs}{2}$ , while the lower solution corresponds to the charged singular vectors  $|\chi_R\rangle_l^{(-1)-}$  at level  $\frac{r(s+2)}{2}$ . The solutions  $\mathbf{h}_k^{BFK} + \frac{1}{2}$  and  $\mathbf{h}_k^{BFK} - \frac{1}{2}$  correspond to the charged singular vectors  $|\chi_R\rangle_l^{(1)+}$  at level  $k$ , and  $|\chi_R\rangle_l^{(-1)-}$  at level  $(-k)$ , respectively.

Therefore, in analogous way as happens for the NS algebra, the zeroes of the vanishing plane  $g_k^P = 0$  give the solutions  $\mathbf{h}_{r,2}^{(1)+}$  and  $\mathbf{h}_{r,2}^{(-1)-}$  (*i.e.*  $\pm \mathbf{h}_{r,2}^{(1)+}$ ), while half of the zeroes of  $f_{r,s}^P = 0$ , for every pair  $(r, s)$ , give the solutions  $\mathbf{h}_{r,s>2}^{(1)+}$  and  $\mathbf{h}_{r,s>2}^{(-1)-}$  (*i.e.*  $\pm \mathbf{h}_{r,s>2}^{(1)+}$ ), corresponding to the charged singular vectors of types  $|\chi_R\rangle_l^{(1)+}$  and  $|\chi_R\rangle_l^{(-1)-}$ . As a result, the complete R Verma modules, with  $\Delta = \frac{c}{24}$  and  $\mathbf{h} = \pm \mathbf{h}_{r,s>2}^{(1)+}$ , contain charged subsingular vectors of types  $|\chi_R\rangle_l^{(1)+}$  and  $|\chi_R\rangle_l^{(-1)-}$ , respectively.

## 5 N=2 Chiral Determinant Formulae

In sections 3 and 4 we have derived conjectures for the roots of the N=2 chiral determinants, corresponding to chiral topological Verma modules, chiral and antichiral NS Verma modules, and R Verma modules built on the R ground states. Using these conjectures, together with some computation and some consistency checks, described below, we have obtained the following expressions for the N=2 chiral determinant formulae.

### *Topological algebra*

The chiral topological Verma modules satisfy  $\Delta = 0$ . The chiral topological determinant formula, in terms of the U(1) charges, is given by the expression:

$$\det(\mathcal{M}_l^T) = \text{cst.} \prod_{2 \leq rs \leq 2l} (\mathbf{h} - \mathbf{h}_{r,s}^{(0)})^{2P_r^T(l - \frac{rs}{2})} (\mathbf{h} - \mathbf{h}_{r,s}^{(1)})^{2P_r^T(l - \frac{rs}{2})} \quad r \in \mathbf{Z}^+, s \in 2\mathbf{Z}^+, \quad (5.1)$$

with the roots  $\mathbf{h}_{r,s}^{(0)}$  and  $\mathbf{h}_{r,s}^{(1)}$ , given by eqns. (3.10) and (3.11), satisfying  $\mathbf{h}_{r,s}^{(1)} = -\mathbf{h}_{r,s}^{(0)} - \mathbf{c}/3$ . The factors 2 in the exponents show that the topological singular vectors in chiral Verma modules come two by two, one charged and one uncharged, at the same level in the same Verma module. Namely, the singular vectors of types  $|\chi_T\rangle_l^{(0)G}$  and  $|\chi_T\rangle_l^{(-1)Q}$  are always together, and the same happens with the singular vectors of types  $|\chi_T\rangle_l^{(1)G}$  and  $|\chi_T\rangle_l^{(0)Q}$ .

The partitions  $P_r^T$  are defined by

$$\sum_N P_r^T(N) x^N = \frac{1}{1+x^r} \sum_n P^T(n) x^n = \frac{1}{1+x^r} \prod_{0 < l \in \mathbf{Z}, 0 < m \in \mathbf{Z}} \frac{(1+x^l)^2}{(1-x^m)^2}. \quad (5.2)$$

### Antiperiodic NS algebra

The chiral and antichiral NS Verma modules satisfy  $\Delta = \pm \mathbf{h}/2$ , respectively. The chiral and antichiral NS determinant formulae are given by a single expression in terms of the conformal weights:

$$\det(\mathcal{M}_l^{NS-ch}) = \det(\mathcal{M}_l^{NS-a}) = \text{cst.} \prod_{2 \leq rs \leq 2l} (\Delta - \Delta_{r,s}^{(0)})^{P_r^{NS}(l - \frac{rs}{2})} \prod_{0 < k = \frac{rs-1}{2} \leq l} (\Delta - \Delta_{r,s}^{(1)})^{P_r^{NS}(l-k)}, \quad (5.3)$$

and by two different expressions in terms of the U(1) charges:

$$\det(\mathcal{M}_l^{NS-ch}) = \text{cst.} \prod_{2 \leq rs \leq 2l} (\mathbf{h} + \mathbf{h}_{r,s}^{(0)})^{P_r^{NS}(l - \frac{rs}{2})} \prod_{0 < k = \frac{rs-1}{2} \leq l} (\mathbf{h} + \mathbf{h}_{r,s}^{(1)})^{P_r^{NS}(l-k)} \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+, \quad (5.4)$$

$$\det(\mathcal{M}_l^{NS-a}) = \text{cst.} \prod_{2 \leq rs \leq 2l} (\mathbf{h} - \mathbf{h}_{r,s}^{(0)})^{P_r^{NS}(l - \frac{rs}{2})} \prod_{0 < k = \frac{rs-1}{2} \leq l} (\mathbf{h} - \mathbf{h}_{r,s}^{(1)})^{P_r^{NS}(l-k)} \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+, \quad (5.5)$$

where  $\Delta_{r,s}^{(0)} = -\mathbf{h}_{r,s}^{(0)}/2$ ,  $\Delta_{r,s}^{(1)} = -\mathbf{h}_{r,s}^{(1)}/2$ , and  $\mathbf{h}_{r,s}^{(0)}$  and  $\mathbf{h}_{r,s}^{(1)}$  are given by eqns. (3.10) and (3.11), like for the Topological algebra.

The partitions  $P_r^{NS}$  are defined by

$$\sum_N P_r^{NS}(N) x^N = \frac{1}{1+x^r} \sum_n P_r^{NS}(n) x^n = \frac{1}{1+x^r} \prod_{0 < k \in \mathbf{Z} + \frac{1}{2}, \quad 0 < m \in \mathbf{Z}} \frac{(1+x^k)^2}{(1-x^m)^2}. \quad (5.6)$$

### Periodic R algebra

The R Verma modules built on R ground states satisfy  $\Delta = \frac{\mathbf{c}}{24}$ . The corresponding R determinant formula, in terms of the U(1) charges, is given by the expression:

$$\det(\mathcal{M}_l^R) = \text{cst.} \prod_{2 \leq rs \leq 2l} (\mathbf{h} - \mathbf{h}_{r,s}^{(0)+})^{2P_r^R(l - \frac{rs}{2})} (\mathbf{h} - \mathbf{h}_{r,s}^{(1)+})^{2P_r^R(l - \frac{rs}{2})} \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+, \quad (5.7)$$

with the roots  $\mathbf{h}_{r,s}^{(0)+}$  and  $\mathbf{h}_{r,s}^{(1)+}$ , given by eqns. (4.7) and (4.8), satisfying  $\mathbf{h}_{r,s}^{(1)+} = -\mathbf{h}_{r,s}^{(0)+}$ . The factors 2 in the exponents show that the R singular vectors built on R ground states

come two by two, one charged and one uncharged, at the same level in the same Verma module. That is, the singular vectors of types  $|\chi_R\rangle_l^{(0)+}$  and  $|\chi_R\rangle_l^{(-1)-}$  on the one hand, and the singular vectors of types  $|\chi_R\rangle_l^{(1)+}$  and  $|\chi_R\rangle_l^{(0)-}$ , on the other hand, are always together. The partitions  $P_r^R$  coincide exactly with the partitions corresponding to the Topological algebra, *i.e.*  $P_r^R = P_r^T$ , defined in equation (5.2). Therefore the exponents are identical for the chiral topological determinants (5.1) as for the R determinants (5.7).

This fact is easy to understand taking into account that the deformation of the chiral and antichiral NS Verma modules into chiral topological Verma modules, under the twists  $T_{W2}$  (2.3) and  $T_{W1}$  (2.2) respectively, follows the same pattern as the deformation of the NS Verma modules into R Verma modules under the spectral flows. In particular, the reorganization of states for the corresponding Verma modules, at every level, satisfies  $l^T = l^R = l^{NS} + \frac{1}{2}|q^{NS}|$ , so that the number of states in the topological Verma modules is equal to the number of states in the R Verma modules level by level.

### Consistency Checks

We have performed consistency checks on the N=2 chiral determinant formulae<sup>||</sup>. They are based on the fact that, for the NS algebra and for the R algebra, the terms of the chiral determinants with highest power of  $\mathbf{c}$  are the diagonal terms (similar statements cannot be made for  $\Delta$  or  $\mathbf{h}$ ).

Let us start with the R algebra, which is the easiest case. A simple exercise shows that the power of  $\mathbf{c}$  in the diagonal of the level  $l$  matrix, for Verma modules built on R ground states, is given by the coefficients of the generating function

$$\prod_{0 < m \in \mathbf{Z}} \frac{(1+x^m)^2}{(1-x^m)^2} \sum_{0 < n \in \mathbf{Z}} \frac{4x^n}{1-x^{2n}} \quad . \quad (5.8)$$

These coefficients are to be compared with the highest power of  $\mathbf{c}$  predicted by the determinant formula (5.7), that is

$$4 \sum_{2 \leq rs \leq 2l} P_r^R \left( l - \frac{rs}{2} \right), \quad (5.9)$$

where one takes into account that  $\mathbf{h}_{r,s}^{(0)+}$  and  $\mathbf{h}_{r,s}^{(1)+}$  contribute with one  $\mathbf{c}$  each. One obtains that the coefficients of the generating function (5.8) can be expressed exactly as (5.9), after a rather straightforward manipulation, using the definitions in (5.2).

One finds analogous results, although more laborious, for the NS algebra. The power of  $\mathbf{c}$  in the diagonal of the level  $l$  matrix for chiral Verma modules is given by the coefficients of the generating function

---

<sup>||</sup>suggested to us by Adrian Kent



$$\frac{1}{(1+x^{1/2})} \prod_{0 < n \in \mathbf{Z}} \frac{1}{(1-x^n)^2} \prod_{0 < k \in \mathbf{Z}+1/2} (1+x^k)^2 \left\{ \sum_{0 < l \in \mathbf{Z}} \frac{2x^l}{1-x^l} - \frac{x}{1-x} + \sum_{\frac{1}{2} < k \in \mathbf{Z}+1/2} \frac{2x^k}{1+x^k} \right\}. \quad (5.10)$$

These coefficients coincide with the highest power of  $\mathbf{c}$  predicted by the determinant formulae (5.3)-(5.5), that is

$$\sum_{2 \leq rs \leq 2l} P_{r+1/2}^{NS}(l - \frac{rs}{2}) + \sum_{0 < k = \frac{rs-1}{2} \leq l, r > 1} P_{r-1/2}^{NS}(l - k), \quad (5.11)$$

where one takes into account that  $\mathbf{h}_{r,s}^{(0)}$  contributes with one  $\mathbf{c}$  each and  $\mathbf{h}_{r,s}^{(1)}$  contributes with one  $\mathbf{c}$  except for  $r = 1$ .

For the Topological algebra there is no need to make an independent consistency check. The reason is that, on the one hand, the roots of the chiral topological determinants are equal to the roots of the antichiral NS determinants, as we showed in section 2, whereas, on the other hand, the exponents of the topological determinants are equal to the exponents of the R determinants, as we just discussed.

## 6 Conclusions and Final Remarks

*First*, we have analyzed in much detail the relation between the singular vectors of the Topological algebra and the singular vectors of the NS algebra, showing the direct relation between their corresponding spectra. Then we have deduced, using the family structure of the topological singular vectors, that charged and uncharged NS singular vectors in chiral NS Verma modules come in pairs, although in different Verma modules, with a precise relation between them:  $V_{NS}^{ch}(\mathbf{h}) \leftrightarrow V_{NS}^{ch}(-\mathbf{h} + \mathbf{c}/3)$  and  $V_{NS}^a(\mathbf{h}) \leftrightarrow V_{NS}^a(-\mathbf{h} - \mathbf{c}/3)$  for chiral and antichiral NS Verma modules, respectively. This result contrasts drastically the case of complete Verma modules, where the charged singular vectors, which correspond to a one-parameter family of roots of the NS determinant formula, are much less numerous than the uncharged singular vectors, which correspond to a two-parameter family of roots of the NS determinant formula. In addition, we have shown that the charged NS singular vectors built on chiral primaries have always relative charge  $q = -1$ , while those built on antichiral primaries have always relative charge  $q = 1$ . These vectors are mirrored to each other under the interchange  $H_m \leftrightarrow -H_m$ ,  $G_r^+ \leftrightarrow G_r^-$  because they are the two possible untwistings of the same topological singular vectors of type  $|\chi_T\rangle^{(1)G}$ .

*Second*, using the result that charged and uncharged singular vectors in chiral Verma modules come in pairs, we have derived a conjecture for the roots of the chiral determinant formulae for the Topological algebra and the NS algebra. For this purpose we have also made the ansatz that the roots of the chiral determinant formulae coincide with the roots of the determinant formulae specialized to the values of (or relations between) the conformal weights and U(1) charges which occur in chiral Verma modules ( $\Delta = 0$  for chiral topological Verma modules and  $\Delta = \pm \mathbf{h}/2$  for chiral and antichiral NS Verma modules, respectively). Our results imply that both uncharged and charged singular vectors, correspond to two-parameter families of roots of the chiral determinant formulae, denoted as  $\mathbf{h}_{r,s}^{(0)}$  eq. (3.10), and  $\mathbf{h}_{r,s}^{(1)}$  eq. (3.11), respectively, and agree with all known data (spectrum of topological and NS singular vectors in chiral Verma modules from level 1/2 to level 4).

Our results also imply the existence of subsingular vectors in the complete Verma modules. These are the singular vectors which are singular only in the chiral Verma modules, becoming non-highest weight null vectors in the complete Verma modules, which are not descendants of any singular vectors. We have found that there are *charged subsingular* vectors, at levels  $\frac{rs-1}{2}$ , in the complete NS Verma modules with  $\Delta = \mp \mathbf{h}/2$  for  $\mathbf{h} = \pm \mathbf{h}_{r,s>2}^{(1)}$ , which contain uncharged singular vectors at levels  $\frac{r(s-2)}{2}$ . Once the chirality is imposed, what amounts to “divide” the complete NS Verma modules by the singular vectors  $G_{-1/2}^-|\Delta, \mathbf{h}\rangle$ , with  $\Delta = -\mathbf{h}/2$ , or  $G_{-1/2}^+|\Delta, \mathbf{h}\rangle$ , with  $\Delta = \mathbf{h}/2$ , the uncharged singular vectors vanish whereas the charged subsingular vectors become singular in the chiral Verma modules. Observe the asymmetry of rôles between “charged” subsingular vectors, which become singular, and “uncharged” singular vectors, which vanish, once the chirality is switched on. This asymmetry does not exist in the Topological algebra where there are both *charged and uncharged subsingular* vectors, and both uncharged and charged singular vectors which vanish when the chirality is imposed and the subsingular vectors become singular. The complete symmetry, in the case of the Topological algebra, is reflected in the fact that the topological subsingular vectors are located in complete topological Verma modules with  $\Delta = 0$ ,  $V(|0, \mathbf{h}\rangle^G)$  and  $V(|0, \mathbf{h}\rangle^Q)$ , with  $\mathbf{h} = \mathbf{h}_{r,s>2}^{(1)}$  and  $\mathbf{h} = \mathbf{h}_{r,s>2}^{(0)}$ , respectively. Under the untwistings only the topological h.w. states (primaries and singular vectors) which are  $\mathcal{G}_0$ -closed remain h.w. states of the NS algebra. For this reason, the symmetry between charged and uncharged topological subsingular vectors is broken in the NS algebra.

*Third*, using the spectral flows between the NS algebra and the R algebra we have translated all the results which apply to chiral NS Verma modules, into results which apply to R Verma modules built on Ramond ground states. In particular we have obtained the corresponding conjecture for the roots of the determinant formula corresponding to this type of R Verma modules (for which  $\Delta = c/24$ ). The situation with respect to subsingular vectors for the R algebra is analogous to the situation for the NS algebra.

Namely, we have found *charged subsingular* vectors in complete R Verma modules with  $\Delta = c/24$  and  $h = \pm h_{r,s>2}^{(1)+}$ , given by eq. (4.8).

*Finally*, using some computer exploration as well as the conjectures for the corresponding roots, we have written down expressions for the chiral determinant formulae, *i.e.* for the chiral topological Verma modules, for the chiral and antichiral NS Verma modules and for the R Verma modules built on the Ramond ground states. We have provided consistency checks for these formulae.

It is already remarkable the fact that the roots of the chiral determinants coincide (as far as we can tell) with the roots of the determinants for the specific relations between  $\Delta$  and  $h$ . We have not found any proof for this ansatz. Also remarkable is the fact that, in the chiral Verma modules, only half of the zeroes of the quadratic vanishing surfaces  $f_{r,s} = 0$ , for every pair  $(r, s)$ , correspond to uncharged singular vectors, while the other half of the solutions correspond to charged singular vectors, in contrast with the case of complete Verma modules for which all the solutions to  $f_{r,s} = 0$  correspond to uncharged singular vectors. Although this remarkable behaviour of the roots of the N=2 determinants is due to the existence of subsingular vectors, it seems to us that there exists a “singular vector conservation law” or “singular vector transmutation” when switching on and off chirality on the primary states, with charged and uncharged singular vectors replacing each other.

In the Appendix we have analyzed thoroughly the NS singular (and/or subsingular) vectors  $|\chi_{NS}\rangle_1^{(0)}$ ,  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$  and  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$ . We have written down the h.w. equations, with their solutions, for the cases of the primary states being non-chiral, chiral and antichiral. We also have investigated the behavior of the subsingular vectors, and we have attempted to shed some light on the issue of the “singular vector transmutation”.

As a final remark, we stress the fact that we have discovered subsingular vectors in the N=2 Superconformal algebras, their very existence was completely unknown, and we have written down examples. Subsingular vectors do not exist in the Virasoro algebra neither in the  $sl(2)$  algebra. The subsingular vectors that we have found are those which become singular in the chiral Verma modules. The issue whether these are the only subsingular vectors in the N=2 Superconformal algebras is currently under investigation.

### Acknowledgements

We thank very much A. Kent for many explanations and suggestions about determinant formulae and related issues. We also thank very much A.N. Schellekens for many illuminating discussions; his insight was essential for the discovery of subsingular vectors.

We also thank the Tata Institute of Fundamental Research (Bombay) for hospitality, where an important part of this paper was worked out. In particular we thank S. Mukhi and A.K. Raina for useful comments to this work. We are indebted to J. Gaiete, A.N. Schellekens, Ch. Schweigert, and especially J. Vermaseren for help with the computer. The chiral determinants have been computed using FORM. Finally, we are very grateful to the referee of Nuclear Physics for reading our work with so much interest and giving us many useful suggestions to improve our paper (see the note below).

## Important Note

This is an extended and improved version of the work presented last year in hep-th/9602166. Following the suggestions of the referee of Nuclear Physics B: i) we have derived conjectures for the whole  $N=2$  chiral determinant formulae, not only for the roots, and ii) we have given appropriate emphasis to the importance of the discovery of subsingular vectors for the  $N=2$  Superconformal algebras. Between the first version and the present version we have published some examples of subsingular vectors for the Topological algebra in ref [11]. To our knowledge no other authors have given examples of subsingular vectors for the  $N=2$  Superconformal algebras. Recently a paper has appeared by A.M. Semikhatov and I.Yu. Tipunin (hep-th/9704111), dealing with  $N=2$  subsingular vectors, in which there are many claims for which no proofs are given. Furthermore we disagree with several statements.

## Appendix

In what follows we give a very detailed example of the differences between singular vectors in chiral NS Verma modules and singular vectors in complete NS Verma modules. We present the highest weight (h.w.) conditions, with their solutions, which define the singular vectors  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$ ,  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$  and  $|\chi_{NS}\rangle_1^{(0)}$ , for the cases when the primary state  $|\Delta, \mathbf{h}\rangle$  is: i) non-chiral, indicating the results for  $\Delta = \pm \mathbf{h}/2$ , ii) chiral, *i.e.* which satisfy  $G_{-1/2}^+|\Delta, \mathbf{h}\rangle = 0$ ,  $\Delta = \mathbf{h}/2$ , and iii) antichiral, *i.e.* which satisfy  $G_{-1/2}^-|\Delta, \mathbf{h}\rangle = 0$ ,  $\Delta = -\mathbf{h}/2$ . The subsingular vectors are the states which are singular in the chiral or antichiral Verma modules, but not in the complete Verma modules. We show that these states, which correspond to the roots  $\pm \hat{\mathbf{h}}_{r,s}$  in (3.4), or equivalently  $\pm \mathbf{h}_{r,s>2}^{(1)}$  in (3.11), are non-h.w. null vectors in the complete Verma modules, not descendants of any singular vectors.

*The singular/subsingular vectors  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$*

The general form of the charge  $q = 1$  NS singular vectors at level  $\frac{3}{2}$  is

$$|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)} = (\alpha L_{-1} G_{-1/2}^+ + \beta H_{-1} G_{-1/2}^+ + \gamma G_{-3/2}^+) |\Delta, \mathbf{h}\rangle. \quad (\text{A.1})$$

The h.w. conditions  $L_{m>0}|\chi\rangle = H_{m>0}|\chi\rangle = G_{r\geq\frac{1}{2}}^+|\chi\rangle = G_{r\geq\frac{1}{2}}^-|\chi\rangle = 0$ , which determine the coefficients  $\alpha, \beta, \gamma$ , as well as the conformal weight  $\Delta$  and the U(1) charge  $\mathbf{h}$  of the primary state, result as follows. For  $|\Delta, \mathbf{h}\rangle$  non-chiral one obtains the equations

$$\begin{aligned} \alpha(1 + 2\Delta) + \beta(1 + \mathbf{h}) + 2\gamma &= 0 \\ \alpha(1 + \mathbf{h}) + \beta \frac{\mathbf{c}}{3} + \gamma &= 0 \\ (2\alpha + \beta)(2\Delta - \mathbf{h}) + \gamma(2\Delta - 3\mathbf{h} + 2\frac{\mathbf{c}}{3}) &= 0 \\ \beta(2\Delta - \mathbf{h}) - 2\gamma &= 0 \\ \alpha(2\Delta - \mathbf{h}) + 2\gamma &= 0 \\ \alpha + \beta &= 0. \end{aligned} \quad (\text{A.2})$$

We see that, for  $\Delta \neq \frac{\mathbf{h}}{2}$ ,  $\gamma$  must necessarily be different from zero (if  $\gamma = 0$  the whole vector vanishes). Hence we can choose  $\gamma = 1$ . Solving for the other coefficients one obtains the solution, for  $\Delta \neq \frac{\mathbf{h}}{2}$

$$\alpha = \frac{-2}{2\Delta - \mathbf{h}}, \quad \beta = \frac{2}{2\Delta - \mathbf{h}} \quad (\text{A.3})$$

with  $\Delta - \frac{3}{2}\mathbf{h} + \frac{\mathbf{c}-3}{3} = 0$ . This solution is given by the vanishing plane  $g_{3/2} = 0$ , as one can check in eq. (3.2).

For the case  $\Delta = \frac{\mathbf{h}}{2}$  the solution is  $\gamma = 0$ ,  $\beta = -\alpha$  and  $\mathbf{h} = \frac{\mathbf{c}-3}{3}$ . It is also given by the vanishing plane  $g_{3/2} = 0$ . If we now specialize the general solution (A.3) to the case  $\Delta = -\frac{\mathbf{h}}{2}$  we find

$$\alpha = \frac{6}{\mathbf{c} - 3}, \quad \beta = \frac{6}{3 - \mathbf{c}}, \quad \mathbf{h} = \frac{\mathbf{c} - 3}{6}. \quad (\text{A.4})$$

For  $|\Delta, \mathbf{h}\rangle$  *chiral* the h.w. conditions only give the equation  $\gamma = 0$ . Therefore  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)ch}$  vanishes while  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)}$ , for  $\Delta = \frac{\mathbf{h}}{2}$ , is a singular vector with  $\gamma = 0$ ,  $\beta = -\alpha$ , as we have just shown. This is a particular case of the general result that the charged singular vectors built on chiral primaries have always relative charge  $q = -1$ , whereas those built on antichiral primaries have always  $q = 1$ .

For  $|\Delta, \mathbf{h}\rangle$  *antichiral* one gets the equations

$$\begin{aligned}
\alpha(1 - \mathbf{h}) + \beta(1 + \mathbf{h}) + 2\gamma &= 0 \\
\alpha(1 + \mathbf{h}) + \beta \frac{\mathbf{c}}{3} + \gamma &= 0 \\
(2\alpha + \beta)\mathbf{h} + \gamma(2\mathbf{h} - \frac{\mathbf{c}}{3}) &= 0 \\
\beta \mathbf{h} + \gamma &= 0 \\
\alpha(1 - \mathbf{h}) + \beta + \gamma &= 0.
\end{aligned} \tag{A.5}$$

Comparing these with equations (A.2), setting  $\Delta = -\frac{\mathbf{h}}{2}$ , we see that here there is one equation less and the first four equations coincide. The last equation here and the two last equations in (A.2) are different though. These equations correspond to the h.w. condition  $G_{\frac{1}{2}}^- |\chi_{NS}\rangle_{\frac{3}{2}}^{(1)} = 0$ . As before  $\gamma \neq 0$  necessarily, thus we set  $\gamma = 1$ . Solving for the other coefficients and for  $\mathbf{h}$  one obtains two solutions:

$$\alpha = \begin{cases} \frac{6}{\mathbf{c}-3} \\ \frac{\mathbf{c}-3}{6} \end{cases}, \quad \beta = \begin{cases} \frac{6}{3-\mathbf{c}} \\ -1 \end{cases}, \quad \mathbf{h} = \begin{cases} \frac{\mathbf{c}-3}{6} \\ 1 \end{cases}. \tag{A.6}$$

The solution  $\mathbf{h} = \frac{\mathbf{c}-3}{6}$  is the solution given by the vanishing plane  $g_{3/2} = 0$ , and therefore the only solution for  $\Delta = -\frac{\mathbf{h}}{2}$ ,  $|\Delta, \mathbf{h}\rangle$  non-chiral, as we have shown in eq. (A.4). It corresponds to  $\mathbf{h}_{2,2}^{(1)}$  in eq. (3.11). The solution  $\mathbf{h} = 1$  corresponds to  $\hat{\mathbf{h}}_{1,2}$  in eq. (3.4), and to  $\mathbf{h}_{1,4}^{(1)}$  in eq. (3.11), given by the vanishing quadratic surface  $f_{1,2} = 0$ . We see therefore that the charged subsingular vector (in the complete Verma module) corresponds to  $\mathbf{h}_{r,s>2}^{(1)}$  whereas the solution  $\mathbf{h}_{r,2}^{(1)}$ , given by the vanishing plane, corresponds to a singular vector which is also singular in the complete Verma module.

*The singular/subsingular vectors  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)}$*

The general form of the charge  $q = -1$  NS singular vectors at level  $\frac{3}{2}$  is

$$|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)} = (\alpha L_{-1} G_{-1/2}^- + \beta H_{-1} G_{-1/2}^- + \gamma G_{-3/2}^-) |\Delta, \mathbf{h}\rangle. \tag{A.7}$$

Since this case is very similar to the previous one we will consider only the chiral representations. The h.w. conditions result in  $\gamma = 0$  for  $|\Delta, \mathbf{h}\rangle$  antichiral (therefore  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)a}$  vanishes), while for  $|\Delta, \mathbf{h}\rangle$  chiral one gets the equations

$$\begin{aligned}
\alpha(1 + \mathbf{h}) + \beta(\mathbf{h} - 1) + 2\gamma &= 0 \\
\alpha(\mathbf{h} - 1) + \beta \frac{\mathbf{c}}{3} - \gamma &= 0
\end{aligned}$$

$$\begin{aligned}
(2\alpha - \beta)\mathbf{h} + \gamma(2\mathbf{h} + \frac{\mathbf{c}}{3}) &= 0 \\
\beta \mathbf{h} + \gamma &= 0 \\
\alpha(\mathbf{h} + 1) - \beta + \gamma &= 0,
\end{aligned} \tag{A.8}$$

where, as before, we can set  $\gamma = 1$ . The other coefficients and the U(1) charge  $\mathbf{h}$  read

$$\alpha = \begin{cases} \frac{6}{\frac{\mathbf{c}-3}{6}} \\ \frac{\frac{\mathbf{c}-3}{6}}{1} \end{cases}, \quad \beta = \begin{cases} \frac{6}{\frac{\mathbf{c}-3}{6}} \\ 1 \end{cases}, \quad \mathbf{h} = \begin{cases} \frac{3-\mathbf{c}}{6} \\ -1 \end{cases}. \tag{A.9}$$

These solutions correspond to  $(-\mathbf{h}_{2,2}^{(1)})$  and  $(-\mathbf{h}_{1,4}^{(1)})$  respectively. The first solution is given by the vanishing plane  $g_{-3/2} = 0$ , and the second solution, given by the vanishing quadratic surface  $f_{1,2} = 0$ , corresponds to a subsingular vector in the complete Verma module, as in the previous case.

We see that  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$  and  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$  are mirrored under the interchange  $H_m \leftrightarrow -H_m$ ,  $G_r^+ \leftrightarrow G_r^-$ , reflecting the fact that they are the two different untwistings of the same topological singular vectors  $|\chi_T\rangle_2^{(1)G}$ , given by

$$|\chi_T\rangle_2^{(1)G} = (\mathcal{G}_{-2} + \alpha\mathcal{L}_{-1}\mathcal{G}_{-1} + \beta\mathcal{H}_{-1}\mathcal{G}_{-1})|\phi\rangle_{\mathbf{h}} \tag{A.10}$$

with

$$\alpha = \begin{cases} \frac{6}{\frac{\mathbf{c}-3}{6}} \\ \frac{\frac{\mathbf{c}-3}{6}}{1} \end{cases}, \quad \beta = \begin{cases} \frac{6}{\frac{3-\mathbf{c}}{6}} \\ -1 \end{cases}, \quad \mathbf{h} = \begin{cases} \frac{\mathbf{c}-3}{6} \\ 1 \end{cases}. \tag{A.11}$$

as the reader can verify using the twists  $T_{W1}$  (2.2) and  $T_{W2}$  (2.3).

*The singular vectors  $|\chi_{NS}\rangle_1^{(0)}$*

The general form of the uncharged NS singular vectors at level 1 is given by

$$|\chi_{NS}\rangle_1^{(0)} = (\alpha L_{-1} + \beta H_{-1} + \gamma G_{-1/2}^+ G_{-1/2}^-)|\Delta, \mathbf{h}\rangle. \tag{A.12}$$

The h.w. conditions result as follows. For  $|\Delta, \mathbf{h}\rangle$  non-chiral one obtains the equations

$$\begin{aligned}
2\alpha \Delta + \beta \mathbf{h} + \gamma(2\Delta + \mathbf{h}) &= 0 \\
\alpha \mathbf{h} + \beta \frac{\mathbf{c}}{3} + \gamma(2\Delta + \mathbf{h}) &= 0 \\
\alpha - \beta - \gamma(2\Delta + \mathbf{h}) &= 0
\end{aligned}$$

$$\alpha + \beta + \gamma(2\Delta - \mathbf{h} + 2) = 0. \quad (\text{A.13})$$

As before we can set  $\gamma = 1$  and we get

$$\alpha = \mathbf{h} - 1, \quad \beta = -(2\Delta + 1), \quad (\text{A.14})$$

with  $\mathbf{h}^2 - 2\Delta\frac{\mathbf{c}-3}{3} - \frac{\mathbf{c}}{3} = 0$ . This solution corresponds to the quadratic vanishing surface  $f_{12} = 0$  in (3.1). It has been given before in refs. [24] and [25].

For  $\Delta = \frac{\mathbf{h}}{2}$  the solutions are

$$\alpha = \begin{Bmatrix} \frac{\mathbf{c}-3}{3} \\ -2 \end{Bmatrix}, \quad \hat{\alpha} = \begin{Bmatrix} \frac{\mathbf{c}+3}{3} \\ 0 \end{Bmatrix}, \quad \beta = \begin{Bmatrix} -\frac{\mathbf{c}+3}{3} \\ 0 \end{Bmatrix}, \quad \mathbf{h} = \begin{Bmatrix} \frac{\mathbf{c}}{3} \\ -1 \end{Bmatrix}, \quad (\text{A.15})$$

$\alpha$  transforming into  $\hat{\alpha}$  if we commute the term  $G_{-\frac{1}{2}}^+ G_{-\frac{1}{2}}^- \rightarrow G_{-\frac{1}{2}}^- G_{-\frac{1}{2}}^+$ . These solutions correspond to  $(-\mathbf{h}_{1,2})$  and  $(-\hat{\mathbf{h}}_{1,2})$  in eqns. (3.3) and (3.4), respectively.

For  $\Delta = -\frac{\mathbf{h}}{2}$  the solutions are

$$\alpha = \begin{Bmatrix} -\frac{\mathbf{c}+3}{3} \\ 0 \end{Bmatrix}, \quad \beta = \begin{Bmatrix} -\frac{\mathbf{c}+3}{3} \\ 0 \end{Bmatrix}, \quad \mathbf{h} = \begin{Bmatrix} -\frac{\mathbf{c}}{3} \\ 1 \end{Bmatrix}. \quad (\text{A.16})$$

These solutions correspond to  $\mathbf{h}_{1,2}$  and  $\hat{\mathbf{h}}_{1,2}$  in eqns. (3.3) and (3.4), respectively.

For  $|\Delta, \mathbf{h}\rangle$  *chiral* we can set  $\gamma = 0$  and the h.w. conditions on  $|\chi_{NS}\rangle_1^{(0)ch}$  give the equations

$$\begin{aligned} \alpha + \beta &= 0 \\ \alpha \mathbf{h} + \beta \frac{\mathbf{c}}{3} &= 0. \end{aligned} \quad (\text{A.17})$$

Comparing these with eqns. (A.13) for  $\Delta = \frac{\mathbf{h}}{2}$ , we see that the first and fourth equations in (A.13) coincide now with the first one here, while the third equation in (A.13), which corresponds to the h.w. condition  $G_{\frac{1}{2}}^+ |\chi\rangle = 0$ , has disappeared, the reason being that the complete equation reads  $(\alpha - \beta - \gamma(2\Delta + \mathbf{h}))G_{-\frac{1}{2}}^+ |\Delta, \mathbf{h}\rangle = 0$ . The solution to these equations is  $\beta = -\alpha$ ,  $\mathbf{h} = \frac{\mathbf{c}}{3}$ , that is

$$|\chi_{NS}\rangle_1^{(0)ch} = (L_{-1} - H_{-1}) |\Delta = \mathbf{c}/6, \mathbf{h} = \mathbf{c}/3\rangle. \quad (\text{A.18})$$



Therefore, only the solution  $(-\mathbf{h}_{1,2}) = \frac{\mathbf{c}}{3}$ , denoted as  $(-\mathbf{h}_{1,2}^{(0)})$  in eq. (3.10), remains after switching on chirality on the primary state  $|\Delta, \mathbf{h}\rangle$ , whereas the singular vector  $|\chi_{NS}\rangle_1^{(0)}$  vanishes for  $(-\hat{\mathbf{h}}_{1,2}) = -1$ , as one can check in (A.15).

For  $|\Delta, \mathbf{h}\rangle$  antichiral the h.w. conditions on  $|\chi_{NS}\rangle_1^{(0)a}$  give the equations

$$\begin{aligned}\alpha - \beta &= 0 \\ \alpha \mathbf{h} + \beta \frac{\mathbf{c}}{3} &= 0.\end{aligned}\tag{A.19}$$

Comparing these with eqns. (A.13) we see that now the last equation in (A.13), which corresponds to the h.w. condition  $G_{\frac{1}{2}}^-|\chi\rangle = 0$ , has disappeared, the reason being that the complete equation reads  $(\alpha + \beta + \gamma(2\Delta - \mathbf{h} + 2))G_{-\frac{1}{2}}^-|\Delta, \mathbf{h}\rangle = 0$ . The solution to these equations is  $\beta = \alpha$ ,  $\mathbf{h} = -\frac{\mathbf{c}}{3}$ , that is

$$|\chi_{NS}\rangle_1^{(0)a} = (L_{-1} + H_{-1}) |\Delta = \mathbf{c}/6, \mathbf{h} = -\mathbf{c}/3\rangle.\tag{A.20}$$

Therefore, only the solution  $\mathbf{h}_{1,2} = -\frac{\mathbf{c}}{3}$ , denoted as  $\mathbf{h}_{1,2}^{(0)}$  in eq. (3.10), remains after switching on antichirality on the primary state  $|\Delta, \mathbf{h}\rangle$ , whereas the singular vector  $|\chi_{NS}\rangle_1^{(0)}$  vanishes for  $\hat{\mathbf{h}}_{1,2} = 1$ , as one can check in (A.16).

Observe that  $|\chi_{NS}\rangle_1^{(0)ch}$  and  $|\chi_{NS}\rangle_1^{(0)a}$  are symmetric under the interchange  $H_m \leftrightarrow -H_m$ ,  $G_r^+ \leftrightarrow G_r^-$ , as expected. This also happens for the singular vectors  $|\chi_{NS}\rangle_1^{(0)}$  specialized to the cases  $\Delta = \frac{\mathbf{h}}{2}$  and  $\Delta = -\frac{\mathbf{h}}{2}$ , eqns. (A.15) and (A.16) (one has to take into account that the commutation  $G_{-1/2}^+ G_{-1/2}^- \rightarrow G_{-1/2}^- G_{-1/2}^+$  produces a global minus sign). As a matter of fact, the singular vectors  $|\chi_{NS}\rangle_1^{(0)ch}$  and  $|\chi_{NS}\rangle_1^{(0)a}$  are nothing but the singular vectors  $|\chi_{NS}\rangle_1^{(0)}$  given by the solutions (A.15) and (A.16) (for  $\mathbf{h} = \pm\mathbf{c}/3$ ) after dropping the term  $G_{-1/2}^+ G_{-1/2}^-$  due to the chirality and antichirality conditions on  $|\Delta, \mathbf{h}\rangle$ .

### *Subsingular vector behaviour*

Now let us see that the charged subsingular vectors at level  $\frac{3}{2}$  behave properly; that is, they are not descendant null vectors of the uncharged singular vectors which sit at level 1 in the complete Verma modules. As a matter of fact, we will see that these charged subsingular vectors, which are null, can descend down to the uncharged singular vectors, but not the other way around.

To see this in some detail let us write the uncharged singular vectors at level 1 for  $\mathbf{h} = \pm 1$ ,  $\Delta = \mp \frac{\mathbf{h}}{2} = -\frac{1}{2}$ , given by eqns. (A.16) and (A.15). For  $\mathbf{h} = 1$ ,  $\Delta = -\frac{1}{2}$ , the singular vector is

$$|\chi_{NS}\rangle_1^{(0)} = G_{-1/2}^+ G_{-1/2}^- |\Delta = -\frac{1}{2}, \mathbf{h} = 1\rangle. \quad (\text{A.21})$$

Its level  $\frac{3}{2}$  secondary with relative charge  $q = 1$  vanishes since  $G_{-1/2}^+ |\chi_{NS}\rangle_1^{(0)} = 0$ . Similarly, for  $\mathbf{h} = -1$ ,  $\Delta = -\frac{1}{2}$ , the singular vector is

$$|\chi_{NS}\rangle_1^{(0)} = G_{-1/2}^- G_{-1/2}^+ |\Delta = -\frac{1}{2}, \mathbf{h} = -1\rangle, \quad (\text{A.22})$$

and its level  $\frac{3}{2}$  descendant with relative charge  $q = -1$  vanishes. (It is also straightforward to see that the uncharged singular vectors (A.21) and (A.22) vanish when one imposes antichirality and chirality on the primary states, respectively, as we said before.)

On the other hand, the charged singular vectors for  $\mathbf{h} = \pm 1$ ,  $\Delta = \mp \frac{\mathbf{h}}{2} = -\frac{1}{2}$ , given by (A.6) and (A.9), are

$$|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a} = \left(\frac{\mathbf{c}-3}{6} L_{-1} G_{-1/2}^+ - H_{-1} G_{-1/2}^+ + G_{-3/2}^+\right) |\Delta = -\frac{1}{2}, \mathbf{h} = 1\rangle \quad (\text{A.23})$$

and

$$|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch} = \left(\frac{\mathbf{c}-3}{6} L_{-1} G_{-1/2}^- + H_{-1} G_{-1/2}^- + G_{-3/2}^-\right) |\Delta = -\frac{1}{2}, \mathbf{h} = -1\rangle \quad (\text{A.24})$$

If we now switch off antichirality (chirality) on the primary state  $|\Delta, \mathbf{h}\rangle$ , then the h.w. condition  $G_{1/2}^- |\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a} = 0$  ( $G_{1/2}^+ |\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch} = 0$ ) is not satisfied anymore, although the vectors are still null, *i.e.* have zero norm, as the reader can verify. However, these vectors cannot be descendant states of the uncharged singular vectors (A.21) and (A.22), as we have just discussed, nor are they descendant states of any level  $\frac{1}{2}$  singular vectors.

Interesting enough, these level  $\frac{3}{2}$  subsingular vectors do descend to the level 1 uncharged singular vectors (A.21) and (A.22) under the action of  $G_{1/2}^-$  and  $G_{1/2}^+$ , respectively, as is easy to check, but not the other way around. The reason is that the singular vectors (A.21) and (A.22) do not build complete Verma modules, as is the usual case for the singular vectors of the N=2 superconformal algebra.

### *Singular vector “transmutation”*

We have seen that the uncharged singular vectors  $|\chi_{NS}\rangle_1^{(0)}$ , for  $\mathbf{h} = \pm \hat{\mathbf{h}}_{1,2} = \pm 1$ ,  $\Delta = -\frac{1}{2}$ , vanish when one switches on antichirality and chirality, respectively, on  $|\Delta, \mathbf{h}\rangle$ , while the charged subsingular vectors become singular vectors:  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$  for  $\mathbf{h} = \hat{\mathbf{h}}_{1,2} = 1$  and  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$  for  $\mathbf{h} = -\hat{\mathbf{h}}_{1,2} = -1$ . The other way around when one switches off

antichirality or chirality on  $|\Delta, \mathbf{h}\rangle$ ; that is, the uncharged singular vectors appear and the charged singular vectors “disappear” (they are not singular vectors anymore), for the values of  $\mathbf{h}$  indicated before, as if a mechanism of “singular vector conservation” or “transmutation” underlies the process.

To shed more light on the issue let us analyze the behaviour of the uncharged singular vector  $|\chi_{NS}\rangle_1^{(0)}$  and its level  $\frac{3}{2}$  descendants near the limits  $\mathbf{h} \rightarrow \pm 1$ ,  $\Delta \rightarrow -\frac{1}{2}$ .

Let us start with  $\mathbf{h}$  near 1. Thus we set  $\mathbf{h} = 1 + \epsilon$ ,  $\Delta = -\frac{1}{2}(1 + \delta)$ ,  $\Delta$  and  $\mathbf{h}$  satisfying the quadratic vanishing surface relation, which results in

$$\frac{c}{3} = \frac{\delta - 2\epsilon - \epsilon^2}{\delta}. \quad (\text{A.25})$$

The vector  $|\chi_{NS}\rangle_1^{(0)}$  is expressed now as

$$|\chi_{NS}\rangle_1^{(0)} = (\epsilon L_{-1} + \delta H_{-1} + G_{-1/2}^+ G_{-1/2}^-) |\Delta, \mathbf{h}\rangle. \quad (\text{A.26})$$

Its charge  $q = 1$  descendant at level  $\frac{3}{2}$ , which we denote as  $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$ , is a null vector which, in principle, is not h.w.

$$|\Upsilon\rangle_{\frac{3}{2}}^{(1)} = G_{-1/2}^+ |\chi_{NS}\rangle_1^{(0)} = (\epsilon L_{-1} G_{-1/2}^+ + \delta H_{-1} G_{-1/2}^+ - \delta G_{-3/2}^+) |\Delta, \mathbf{h}\rangle. \quad (\text{A.27})$$

Now comes a subtle point. In principle we can normalize  $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$  in the same way as  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ , *i.e.* dividing all the coefficients by  $(-\delta)$  so that the coefficient of  $G_{-3/2}^+$  is 1, resulting in

$$|\Upsilon\rangle_{\frac{3}{2}}^{(1)} = ((-\frac{\epsilon}{\delta}) L_{-1} G_{-1/2}^+ - H_{-1} G_{-1/2}^+ + G_{-3/2}^+) |\Delta, \mathbf{h}\rangle. \quad (\text{A.28})$$

However, these two normalizations are not equivalent when taking the limit ( $\mathbf{h} = 1, \Delta = -\frac{1}{2}$ ), *i.e.* ( $\epsilon \rightarrow 0, \delta \rightarrow 0$ ). Namely  $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$  vanishes with the first normalization (A.27) whereas with the second normalization (A.28) it becomes the h.w. singular vector  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ , since  $(-\frac{\epsilon}{\delta})$  turns into  $(\frac{c-3}{6})$ , as can be deduced easily from (A.25).

It seems that the two normalizations distinguish whether the primary state  $|\Delta, \mathbf{h}\rangle$  approaches an antichiral or a non-chiral state as  $\mathbf{h} \rightarrow 1, \Delta \rightarrow -\frac{1}{2}$ . This is indeed true, the reason is that if we normalize  $|\chi_{NS}\rangle_1^{(0)}$  in the same way as its descendant  $|\Upsilon\rangle_{\frac{3}{2}}^{(1)}$ , then in the second normalization, *i.e.* dividing by  $(-\delta)$ , it blows up approaching the limit ( $\epsilon \rightarrow 0, \delta \rightarrow 0$ ) unless the term  $G_{-1/2}^+ G_{-1/2}^-$  goes away, exactly what happens if  $|\Delta, \mathbf{h}\rangle$  is antichiral. But the resulting uncharged vector without the term  $G_{-1/2}^+ G_{-1/2}^-$  is not

a singular vector anymore. The action of  $G_{-1/2}^+$  on this vector results precisely in the charged singular vector  $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ .

We see therefore that, in the limit ( $\mathbf{h} = 1, \Delta = -\frac{1}{2}$ ), the first normalization produces the uncharged singular vector at level 1  $|\chi_{NS}\rangle_1^{(0)}$ , with a vanishing charge  $q = 1$  descendant at level  $\frac{3}{2}$ , whereas the second normalization produces the charge  $q = 1$  singular vector at level  $\frac{3}{2}$   $|\chi_{NS}\rangle_{\frac{3}{2}}^{(1)a}$ .

Repeating this analysis for the case  $\mathbf{h}$  near  $-1$  we find that in the limit ( $\mathbf{h} = -1, \Delta = -\frac{1}{2}$ ), the first normalization produces the uncharged singular vector at level 1  $|\chi_{NS}\rangle_1^{(0)}$ , with a vanishing charge  $q = -1$  descendant at level  $\frac{3}{2}$ , whereas the second normalization produces the charge  $q = -1$  singular vector at level  $\frac{3}{2}$   $|\chi_{NS}\rangle_{\frac{3}{2}}^{(-1)ch}$ .

## References

- [1] M. Ademollo et al., Phys. Lett. B62 (1976) 105; Nucl. Phys. B111 (1976) 77; Nucl. Phys. B114 (1976) 297;  
L. Brink and J.H. Schwarz, Nucl. Phys. B121 (1977) 285;
- [2] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B106 (1981) 63; Phys. Lett. B162 (1985) 295;  
S.D. Mathur and S. Mukhi, Phys. Rev. D 36 (1987) 465; Nucl. Phys. B302 (1988) 130;
- [3] H. Ooguri and C. Vafa, Nucl. Phys. B361 (1991) 469; Nucl. Phys. B367 (1991) 83
- [4] J. Gomis and H. Suzuki, Phys. Lett. B278 (1992) 266;  
N. Marcus, talk at String Theory Workshop, Rome (1992), hep-th/9211059
- [5] B. Gato-Rivera and A. M. Semikhatov, Phys. Lett. B293 (1992) 72, Theor. Mat. Fiz. 95 (1993) 239, Theor. Math. Phys. 95 (1993) 536
- [6] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. B401 (1993) 304
- [7] B. Gato-Rivera and A. M. Semikhatov, Nucl. Phys. B408 (1993) 133
- [8] A.M. Semikhatov, Mod. Phys. Lett. A9 (1994) 1867
- [9] B. Gato-Rivera and J.I. Rosado, Phys. Lett. B346 (1995) 63
- [10] B. Gato-Rivera and J.I. Rosado, Phys. Lett. B369 (1996) 7
- [11] B. Gato-Rivera and J.I. Rosado, "Families of Singular and Subsingular Vectors of the Topological N=2 Superconformal Algebra", hep-th/9701041 (1997)

- [12] W. Boucher, D. Friedan and A. Kent, Phys. Lett. B172 (1986) 316
- [13] S. Nam, Phys. Lett. B172 (1986) 323
- [14] M. Kato and S. Matsuda, Phys. Lett. B184 (1987) 184
- [15] T. Eguchi and S. K. Yang, Mod. Phys. Lett. A5 (1990) 1653
- [16] E. Witten, Commun. Math. Phys. 118 (1988) 411; Nucl. Phys. B340 (1990) 281
- [17] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59
- [18] P. Di Vecchia et al., Phys. Lett. B162 (1985) 327
- [19] A. Schwimmer and N. Seiberg, Phys. Lett. B184 (1987) 191
- [20] W. Lerche, C. Vafa and N. P. Warner, Nucl. Phys. B324 (1989) 427
- [21] E.B. Kiritsis, Phys. Rev. D36 (1987) 3048
- [22] B. Gato-Rivera and J.I. Rosado, Mod. Phys. Lett. A11 (1996) 423
- [23] B. Gato-Rivera, “The Even and the Odd Spectral Flows on the N=2 Superconformal Algebras”, hep-th/9707211 (1997)
- [24] E.B. Kiritsis, Int. J. Mod. Phys A3 (1988) 1871
- [25] M. Yu and H.B. Zheng, Nucl. Phys. B288 (1987) 275
- [26] B. Gato-Rivera and J.I. Rosado, “Families of Singular and Subsingular Vectors of the Antiperiodic and Periodic N=2 Superconformal Algebras”, to appear in hep-th
- [27] A. Kent and M. Dörrzapf, private communication
- [28] B.L. Feigin and D.B. Fuchs, “Representations of the Virasoro algebra” in “Representations of Lie groups and related Topics”, Gordon and Breach, 1990